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History of Mathematics: The Real Numbers - Making them Respectable

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In my last column, I described how, at the cost of some apparent artificiality and seemingly needless complication, the imaginary numbers eventually became respectable. Here I describe the analogous process with the real numbers. Paradoxically, the story of how the real numbers themselves became respectable is more convoluted and the processes involved took thousands rather than hundreds of years. Moreover the topic still causes controversy today (although no one is claiming that the reals aren't respectable). But let us begin at the beginning. The real numbers are made up of two sets: the rational numbers and the irrational ones. In my column for Vol 41, No 2, I described the discovery (by the ancient Greeks) of the irrational numbers.[2]

Just as the imaginary numbers needed justification if it was to be accepted that the usual laws of arithmetic applied to them, so also one needs to demonstrate that the irrational numbers can be manipulated in exactly the same way as the rationals. This was a question never raised in my own early mathematical education, and so when I did come to learn of the incorporation of the irrational numbers into the number-line and the seemingly complicated logic underlying this, I had rather much the same reaction as I did when I learned of Hamilton's approach to the complex numbers (as described in my last column): why all this rigmarole?

That the need existed for a justification of the extension of arithmetic laws to cover the irrationals was perhaps most pointedly made by the mathematician Richard Dedekind (1831–1916), who wrote a paper whose title is most accurately translated as "What are numbers and what should they be?" (Sadly, the official English translation of this paper, under the title "The nature and meaning of numbers", is of poor quality, a point noted in my column in Vol 44, No 1.) In the course of this he pointed out the need to *prove* results such as $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, claiming, perhaps rather pompously, that to the best of his knowledge such results had never been established before.

In what follows, I will describe several approaches to the real numbers, but will concentrate on one in particular. This is the one that will almost certainly be the most

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[2]Nowadays, in everyday discourse, the word "irrational" tends to mean "stupid" or "senseless", but here it simply means that these numbers cannot be expressed as *ratios* of integers. However the link to common usage does mean that an aura of disreputability attaches to them. And the name "surd" given to numbers like $\sqrt{2}$ has a clear linguistic connection with the word "absurd", so there is a relatively clear hint that such numbers are not exactly kosher.

familiar to *Parabola*'s readers. Certainly it was my own and it is described by the Wikipedia entry **Construction of the real numbers** as "[having] the advantage that it is close to the way we are used to thinking of real numbers". I shall begin by looking at the decimal expansions of the different numbers. It was first put forward by the Dutch polymath Simon Stevin (1548–1620),[3] who was an early champion of the decimal system.

So let us begin by looking at the way in which different numbers are represented by decimals. If we are representing a fraction (i.e. a rational number), its decimal equivalent may either terminate (e.g. $\frac{1}{2} = 0.5$) or else repeat a cyclic pattern indefinitely (e.g. $\frac{1}{3} = 0.333...$ or $\frac{1}{7} = 0.142\,857\,142\,857\ ...$). For each rational number, a decimal equivalent may be found by means of a simple division process that readers will be familiar with. It is also true that every recurring (repeating) decimal (i.e. one that exhibits a cyclic pattern) may be converted to a fraction. I will demonstrate the process by means of an example. We have $\frac{19}{28} = 0.678\,571\,428\,571\,428\,571\,428...$. The decimal consists of two parts: an initial non-cyclic portion, 0.678 , and a subsequent cyclic one, $571\,428\,571\,428\ ...$. There are various ways to write such expressions and the one I will choose is $19/28 = 0.678\,\overline{571\,428}$. The bar (or *vinculum*) above the final six digits indicates that these digits repeat: the expression 571428 is termed the repetend. There is a simple way to use the decimal expansion to recover the original fraction.

> For the initial section, represent this as a fraction in the usual way: $0.678 = \frac{678}{1000}$.
>
> To this fraction add another constructed as follows: in the numerator place the repetend itself, and in the denominator put a 9 for each digit of the repetend followed by a 0 for each of the initial digits.
>
> In this instance, this second fraction is $\frac{571\,428}{999999000}$. Thus the total is $\frac{678}{1000} + \frac{571\,428}{999999000} = 0.678\,571\,428\,571\,428\,571\,428... = 0.678\,\overline{571\,428}$, as claimed.

(I leave to the reader the task of demonstrating that this procedure is perfectly general; it makes a nice example in the theory of the geometric progression.)

Thus for every rational number, we may assign a terminating or repeating decimal and each such decimal represents a rational number. A further point is that the terminating decimals are in fact special cases of repeating decimals. Thus, for example, $0.5 = 0.5000...$, with a repetend consisting entirely of zeros. (A minor annoyance is that there are in fact two such repeating decimals for each terminating one; for example, $0.5 = 0.5000... = 0.4999...$. One sometimes finds non-mathematicians who maintain that $0.5000... \neq 0.4999...$. I don't know why they pick on this situation, when they quite happily accept that $0.5 = \frac{1}{2} = \frac{2}{4} = 50\%$, etc. But such is life!)

Thus we can say that the set of rational numbers is precisely the same as the set of recurring decimals. It therefore follows that the non-recurring decimals must represent irrational numbers. Each non-recurring decimal represents an irrational number and every irrational number can be represented by an infinite non-recurring decimal.

[3] Another aspect of Stevin's work was the subject of an earlier column (*Function*, Vol. **22**, Part **5**).

However, this was not how Dedekind saw matters. He saw a real number ($\sqrt{2}$ for example) as a "cut" in the number line, separating it into two distinct classes. Thus in the case of $\sqrt{2}$ we have a class $\mathcal{L}$ (for "lower") comprising the set of all rational numbers r such that $r^2 < 2$ and another class $\mathcal{U}$ (for "upper") comprising the set of all rational numbers r such that $r^2 > 2$. The irrational number ($\sqrt{2}$ in this case, but the principle is the same for any irrational number) *was* the set $\mathcal{L}$ (or $\mathcal{U}$ or both).[4] Thus on Dedekind's approach to the reals, one way to view the irrationals is as infinite sets of rational numbers. Alternatively the number $\sqrt{2}$ itself could be thought of a "separator" that kept these two sets apart.

I first learned of this account in 1957, when I went to The University of Melbourne and was taught first year mathematics by Associate Professor Felix Behrend.[5] I still have my notes from this course, and checking them, I discovered that Behrend used the decimal representation of an irrational number and the idea of a separator. The irrational number was seen as the limit of an infinite sequence of rational approximations. That limit itself, however, is not itself rational. It may however be looked at in the following way. Consider the set of approximations to $\sqrt{2}$: $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ ....$ Each of these numbers is rational and they are all members of the set $\mathcal{L}$ generated by $\sqrt{2}$ as they all lie below the actual value of $\sqrt{2}$. However, if we take any number greater than $\sqrt{2}$, this will bound the set $\mathcal{L}$ and will be what we term an "upper bound". There are of course many such upper bounds, all larger than all the members of the set $\mathcal{L}$. If we look at the set of such upper bounds, we may ask if it has a least member. Intuitively it seems so, although this is not a matter of proof; rather it is incorporated as an axiom, and $\sqrt{2}$ is identified with this "least upper bound" of the set $\mathcal{L}$. The axiom that gives us this identity is now termed "Dedekind's Axiom".

The approaches to the real numbers via the decimal expansion or the Dedekind cut are not the only possibilities. In fact there are quite a lot of others, but pre-eminent among these is an account that uses "Cauchy sequences". A *sequence* is a set of numbers forming a succession: t_1 for the first, t_2 for the second, and so on. Such a sequence is said to *converge* to a limit l if the successive terms become closer and closer to l. The way in which this approach is tested is to pick a (usually very small) number ϵ and check whether or not the difference $|t_n - l| < \epsilon$, whenever n exceeds some number N whose value will depend on that of ϵ. I gave an example in my third column for 2009, in which a sequence $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8}$, etc. converges to the number 2. It never actually gets there but the shortfall may be made arbitrarily small. If we apply this approach to the approximations to $\sqrt{2}$, we aim to define $\sqrt{2}$ as the number toward which the sequence $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ ...$ converges.

However, we can't apply this analysis directly to the case here because the aim is to find a way to *define* the limit l (in this case $\sqrt{2}$) and all we have at our disposal at this stage is the set of rational numbers. This difficulty is overcome by means of a

[4]Dedekind decided on "both". Modern authors tend to choose one or other of the remaining possibilities. It may be demonstrated that it doesn't matter which convention we adopt; they are all equivalent.

[5]Behrend was a German Jew, who fled to Britain to escape Nazi persecution, but was subsequently deported by the British as an "enemy alien". Along with many others with similar stories, he ended up in detention in Australia before sanity prevailed and he was released.

subterfuge. We take terms of the sequence (of rational numbers in this case) and ask if they are getting ever closer to *one another*. The test is whether we can ensure that for every small ϵ we can ensure that $|t_n - t_m| < \epsilon$, whenever n and m both exceed some number N. If this condition is met, the sequence is called a *Cauchy sequence*.[6]

If we adopt as an axiom the principle that every Cauchy sequence has a limit, then the Cauchy sequence of rational approximations defines that irrational limit. Indeed it may be shown that this axiom is equivalent to Dedekind's axiom. The Cauchy sequence axiom is particularly easy to accept for a sequence that homes in on its limit from both sides, as in the case of $1,\ 1.5,\ 1.4,\ 1.42, 1.41,\ 1.413,$ etc. as approximations to $\sqrt{2}$, but it also applies to sequences such as $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142$, etc. that come from just one side.

Many mathematicians prefer the Cauchy sequence account because it leads naturally into more general areas of pure mathematics, to a branch of algebra known as "field theory". However, I will not take this path here, but rather note that all the different approaches to the reals are equivalent to one another. The use of decimal representations, the different versions of the Dedekind cut (the sets $\mathcal{L}$ and/or $\mathcal{U}$, or else the separator limit) and the similar versions of the Cauchy sequence approach all yield mathematically equivalent results. In fact one way to prove this is to show that each is equivalent to the infinite decimal version.

Thus, any one of these approaches (and of the various others that I have not discussed) sufficed to make the irrationals (and thus all the reals) "respectable". Nevertheless many questions remain, and in the rest of this article, I will concentrate on one of these before returning to the question with which I began: how do we know that the usual rules of arithmetic apply to them?

The question I will look at before going back to the start concerns the use of the set $\mathcal{L}$ (say) as the definition of an irrational number. In the case of $\sqrt{2}$, we can say that $\mathcal{L}$ is the set of all rational numbers r such that $r^2 < 2$. However, on the decimal approach, we identify $\sqrt{2}$ as the set $\{1,\ 1.4,\ 1.41,\ 1.414,\ ...\}$, a subset of the set $\mathcal{L}$ and thus apparently different. This matter, however, may be resolved by reference to the work of Dedekind's slightly younger contemporary Georg Cantor (1845–1918). Cantor is best remembered today for his work on infinite sets. I will start with a very simple example. Consider two sets: the set of counting numbers $\{1,\ 2,\ 3,\ 4,\ ...\}$ and the set of even numbers $\{2,\ 4,\ 6,\ 8,\ ...\}$. The second is obviously a subset of the first, and yet on another view, the two sets are equivalent. For each member of the first set, there is a corresponding member of the second *and vice versa*. The sets are said to be in *one-to-one correspondence*. This only works because the sets involved are infinite; if we tried this trick with, let us say, the numbers less than 100, then we would run out of pairings, and leave 50 unpaired counting numbers. This difficulty arises because of the upper limit imposed on the numbers; when the sets are infinite, there is no such upper limit and the one-to-one correspondence can go ahead.

Any set that can be placed in one-to-one correspondence with the counting num-

[6]The name honors Augustin-Louis Cauchy, one of mathematics' all-time greats. Among his many achievements was the establishment of a sound foundation for the calculus. I described some of this in my column in Vol 41, No 1.

bers is said to be *countable*. Cantor was able to demonstrate that the set of rational numbers is countable. It follows that any infinite subset of the rationals is also countable. Any two such sets can thus be placed in one-to-one correspondence with one another. So the set $\{1,\ 1.4,\ 1.414,\ 1.4142,\ ...\}$ and the full set L are equivalent, just as the set of even numbers is equivalent to the set of counting numbers. The members of these two sets may be paired off, so that, on this level, they amount to the same thing!

But now to examine the main question: how do we know that the irrational numbers obey the basic laws of arithmetic? It is this question that occupied the late David Fowler in a paper he named after Dedekind: "Dedekind's Theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ "[7] Dedekind had claimed that results such as this "had never been established before". Fowler would seem to concur, but I think that both Dedekind and he are wrong here. Much of the rest of this paper will be devoted to a critique of Fowler's thesis. I embark on this with a certain reluctance in view of Fowler's eminence. He was one of the twentieth century's best historians of mathematics. I sided with him in his debate with Unguru on the question of mathematical induction (see my column for Vol 49, No 3)[8]. However, in this present paper he was below his best. In particular, he embarked on criticisms (unfair ones in my book) of other authors' work.

For example: " Many mathematicians have a touching and naive belief that arithmetic operations on decimals pose no problems; or they pretend to believe this, as in some circumstances the most scrupulously honest among us may sometimes pretend to believe in Father Christmas ...; or perhaps have never considered the question to be problematic."

The key question is "Can we justify the application of the basic rules of arithmetic to the irrational numbers?" In my column for Vol 50, No 1, I showed that the basic laws of multiplication were obeyed by the natural numbers, and it is only a minor extension of this proof to demonstrate that the rational numbers also obey these rules. I won't stop to prove this here, but it is a point also acknowledged by Fowler. What is at issue is whether or not we can extend the application of these rules to the irrationals. The rules in question are:

> The commutative law of multiplication: $a \times b = b \times a$, and the associative law of multiplication: $a \times (b \times c) = (a \times b) \times c$.

Again, I won't stop to give the formal proof of the next step (which is tedious rather than enlightening), but I will pass on the conclusion: that we may dispense altogether with parentheses, and shuffle the various factors in any way at all without affecting the product.

The key question is this: does the result just announced apply also to the irrational numbers? This is where Fowler raises objections. He shows that if irrational numbers are defined as infinite decimals, then it is difficult, if not impossible, to describe an algorithm (process) that gives the product of two irrational numbers.

[7] Published in *American Mathematical Monthly* in October 1992.

[8] I also have a more personal reason to respect the late Professor Fowler's memory. When some years ago, Monash University sought to dismiss Professor Hans Lausch and myself, Professor Fowler's was one of the influential international voices raised successfully on our behalf.

However, to my mind, he misses the point. The issue is whether the basic laws of multiplication apply to the irrationals. They certainly would if such an algorithm were available, but this is not the only way to approach the problem. A passage I omitted from the "Father Christmas" quote reproduced above directs the reader's attention to a passage in a textbook by Lipman Bers. Bers was a very considerable mathematician, and, as I see things, his account is quite in order. True, he does not dot every *i* and cross every *t*, but the gist of the argument is presented clearly enough.

Very much the same argument was presented in 1821 by Cauchy in his text *Cours d'analyse*. This is also given notice by Fowler, who writes:

"Cauchy's [text] ... has a long appended Note 1 ... in which he defines arithmetical operations on 'numbers' ... in rather vague terms of manipulations of rational approximations ... ."

Here is what Cauchy actually said in relation to the product AB (I quote from the excellent recent translation and commentary by Bradley and Sandifer): "When B is an irrational number, we can obtain rational numbers that approach it more and more closely. We can easily see that under the same hypothesis the product of A by rational numbers in question approaches a limit more and more closely. This limit is the product of A by B."

This is supplemented by the remark that the factors in a product can be addressed in any order: "The product of several quantities remains the same in whatever order we multiply them". In other words, although Cauchy doesn't say it explicitly, the commutative law and the associative laws both hold. The proofs are not given in detail but may readily be supplied from outlines that are. Suppose, for example, that the commutative law were to fail. Then for some pair of irrational numbers A, B we would have $AB \neq BA$, but all the rational approximations (however exact they might be) would obey the law. This is clearly contradictory.

Fowler almost grants as much: " ... vague though his account often is, Cauchy does not fudge the issue by describing arithmetic in terms of terminating decimal expansions, and then pretend that he has described arithmetic in general."

But now, everything that is needed to show that the reals may be manipulated exactly as rational numbers has been shown, and we are able to use ordinary arithmetic in our dealings with the real numbers.

With this behind us, it is easy to prove "Dedekind's Theorem". Here is one ordering of the steps involved.

$$\sqrt{2} \times \sqrt{3} = \sqrt{\left(\left(\sqrt{2} \times \sqrt{3}\right)^2\right)} = \sqrt{\sqrt{2} \times \sqrt{3} \times \sqrt{2} \times \sqrt{3}} =$$
$$\sqrt{\sqrt{2} \times \sqrt{2} \times \sqrt{3} \times \sqrt{3}} = \sqrt{2 \times 3} = \sqrt{6}.$$

All that is used here is the reordering of the various factors, as envisaged by Cauchy.

One final remark is perhaps in order. Irrational numbers themselves fall into two categories: the algebraic and the transcendental. Algebraic irrationals are the roots of polynomial equations with integral coefficients. Clearly $\sqrt{2}$ and $\sqrt{3}$ are algebraic, and it is this that allows the proof of "Dedekind's Theorem". Transcendental irrationals like π are more difficult to deal with.