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Tiling the plane with equilateral convex pentagons

Maria Fischer[1]

Mathematicians and non-mathematicians have been concerned with finding pentagonal tilings for almost 100 years, yet tiling the plane with convex pentagons remains the only unsolved problem when it comes to monohedral polygonal tiling of the plane. One of the oldest and most well known pentagonal tilings is the Cairo tiling shown below. It can be found in the streets of Cairo, hence the name, and in many Islamic decorations.

Figure 1: Cairo tiling

There have been 15 types of such pentagons found so far, but it is not clear whether this is the complete list. We will look at the properties of these 15 types and then find a complete list of equilateral convex pentagons which tile the plane – a problem which has been solved by Hirschhorn in 1983.

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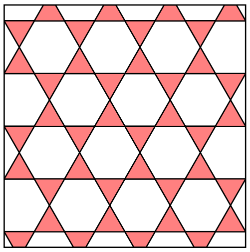
Archimedean/Semi-regular tessellation

An Archimedean or semi-regular tessellation is a regular tessellation of the plane by two or more convex regular polygons such that the same polygons in the same order surround each polygon. All of these polygons have the same side length. There are eight such tessellations in the plane:



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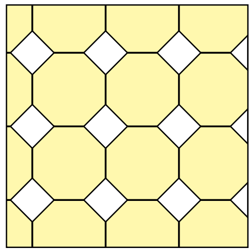

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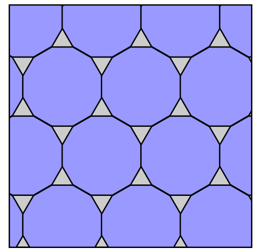

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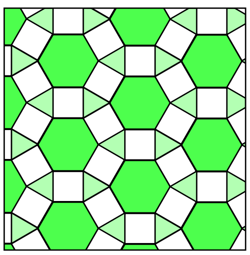

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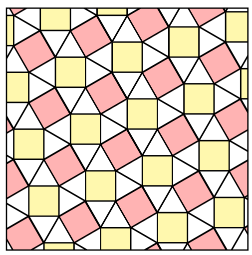

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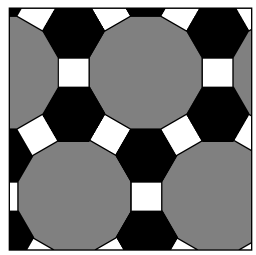

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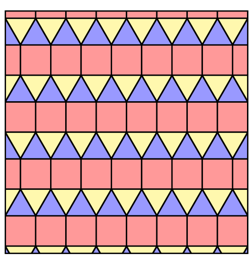

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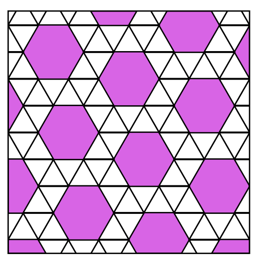

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Number 5 and number 7 involve triangles and squares, number 1 and number 8 involve triangles and hexagons, number 2 involves squares and octagons, number 3 triangles and dodecagons, number 4 involves triangles, squares and hexagons and number 6 squares, hexagons and dodecagons. Number 4 is also one of the many tilings that can be found throughout the Spanish city of Seville. Clearly, Archimedean tessellations are not monohedral.

Monohedral tiling with triangles, quadrilaterals and hexagons

In tiling, we are particularly interested in monohedral tilings, which is a tessellation of the plane in which all tiles are congruent; it has only one prototile.

A triangle is a polygon with three sides and all triangles tessellate. To see that they do, take an arbitrary triangle and rotate it by 180° about the midpoint of one of its sides. It then becomes a parallelogram, which is a polygon with four sides, hence a quadrilateral.

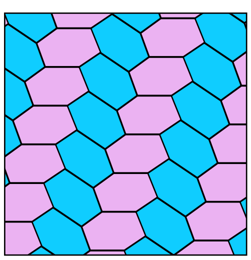
To see that all quadrilaterals tessellate, take an arbitrary quadrilateral with angles A, B, C and D and rotate it by 180° about the midpoint of one of its sides. To build up a tessellation, repeat to rotate about the midpoint of other sides. We observe that the angles around each vertex are exactly A, B, C and D. Since the sum of the angles in a quadrilateral is always 360°, there are no gaps or overlaps, so all quadrilaterals tessellate. Hence, all triangles and quadrilaterals tessellate. Note though, all triangles are convex, but the tessellation with quadrilaterals applies to convex and concave quadrilaterals. We can see that the angles play a key role in tessellation, as they in some way have to add up to 360°.

The regular hexagon also tiles the plane. Its angles are 120° each, so any three angles add up to 360° and hence close up at a vertex (honeycomb pattern). Other than the triangle or quadrilateral, there are only three types of monohedral convex hexagon tilings. With angles A, B, C, D, E and F arranged anticlockwise and sides a, b, c, d, e and f where b for example denotes the side between A and B, the types are the following[2].

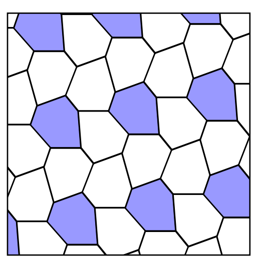
Type 1: $b = e,\ B + C + D = 360^\circ.$

Type 2: $b = e,\ d = f,\ B + C + E = 360^\circ.$

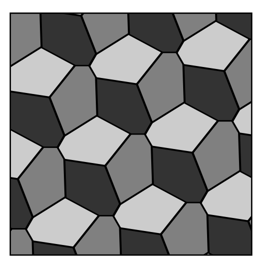
Type 3: $a = f,\ b = c,\ d = e,\ B = D = F = 120^\circ.$



Type 1



Type 2



Type 3

[2]B. Grünbaum and G. C. Shephard, *Tilings and Patterns* (1987), W. H. Freeman and Company, New York, 472–518 (Chapter 9)

Theorem 1. *No polygon with more than* 6 *sides can tessellate.*

PROOF. Let n be the number of sides of the polygon. When we look at the tessellation of the plane with convex tiles, it is easy to observe that at least three tiles have to meet in one vertex. Hence, the average of the angles of these tiles cannot be greater than 120°. We further know that the sum of the inner angles of an $n-$gon is $180\,(n-2)$, hence the average is $\frac{180(n-2)}{n}$. Now, let $n > 6$. Then

$$\frac{180\,(n-2)}{n} = 180\left(1-\frac{2}{n}\right) > 180\left(1-\frac{2}{6}\right) = 120,$$

which is a contradiction. □

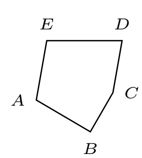
Tiling with pentagons

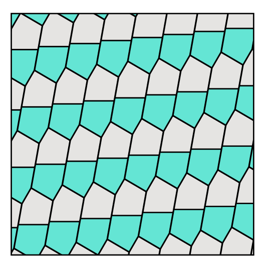
The regular pentagon does not tessellate, as each of the angles is 108° which is not a divisor of 360. However, there are other kinds of pentagons which tile the plane. We call the angles of a pentagon A, B, C, D and E, arranged anticlockwise and the sides a, b, c, d and e where b is the side between A and B and so on. A vertex of one of the polygons is also a vertex of the tiling and we call the valence of a vertex the number of edges at the vertex.

Below is an overview of the tilings which have been discovered so far. They are classified into types and we state when and by whom they have been found. We say that types are distinct if they have different sets of conditions on angles and sides of a pentagon such that each set of conditions is sufficient to ensure that a pentagon with these conditions exists, and that at least one tiling of the plane by such a pentagon exists. Many distinct tilings can exist for pentagons of a given type.

Type 1[3], 1918 Reinhardt

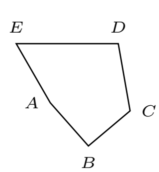
$$D + E = 180°$$

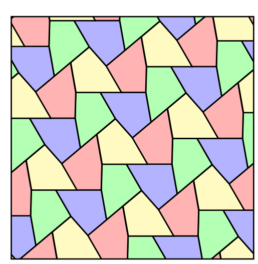




Type 2, 1918 Reinhardt

$$C + E = 180°$$
$$d = a$$

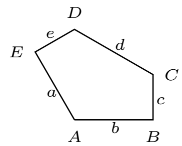


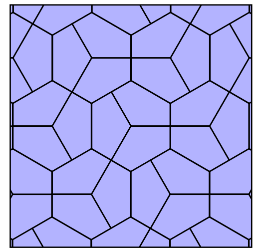


[3] Types 1 – 5 in [K. Reinhardt, Über die Zerlegung der Ebene in Polygone, Dissertation, Universität zu Frankfurt a.M. (1918)]

Type 3, 1918 Reinhardt

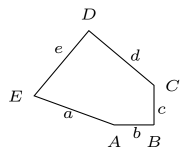
$$A = C = D = 120°$$
$$a = b$$
$$d = c + e$$

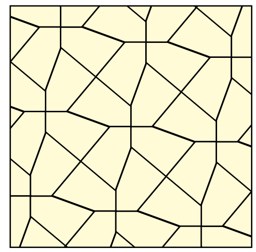




Type 4, 1918 Reinhardt

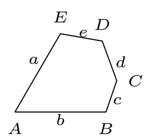
$$B = D = 90°$$
$$b = c$$
$$d = e$$

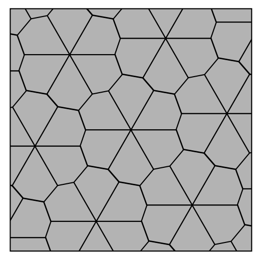




Type 5, 1918 Reinhardt

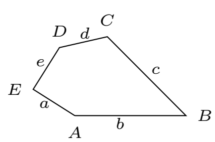
$$A = 60°$$
$$D = 120°$$
$$a = b$$
$$d = e$$

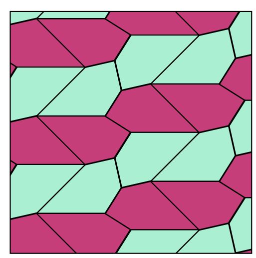




Type 6[4], 1968 Kershner

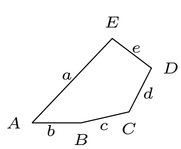
$$B + D = 180°$$
$$2B = E$$
$$a = d = e$$
$$b = c$$

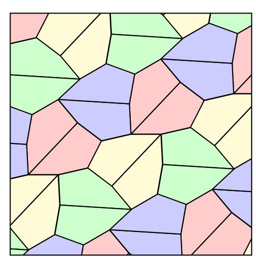




Type 7, 1968 Kershner

$$B + 2E = 360°$$
$$2C + D = 360°$$
$$b = c = d = e$$

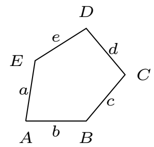


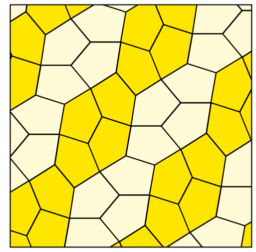


[4]Types 6 – 8 in [R.B. Kershner, On paving the plane, *American Mathematical Monthly* **75** (1968), 839–844]

Type 8, 1968 Kershner

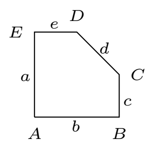
$$2B + C = 360^\circ$$
$$D + 2E = 360^\circ$$
$$b = c = d = e$$

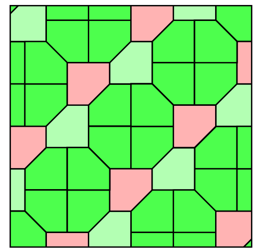




Type 10[5], 1975 James

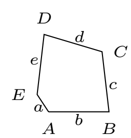
$$A = 90^\circ$$
$$B + E = 180^\circ$$
$$B + 2C = 360^\circ$$
$$a = b = c + e$$

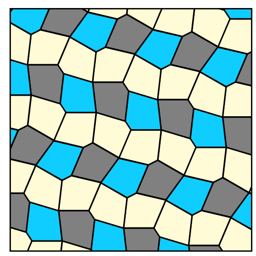




Type 9, 1976 Rice

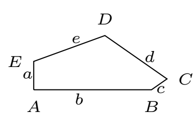
$$2A + C = 360^\circ$$
$$D + 2E = 360^\circ$$
$$b = c = d = e$$

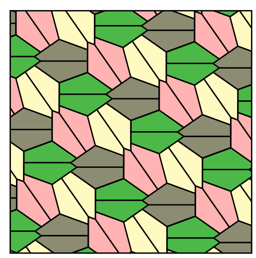




Type 11, 1976 Rice

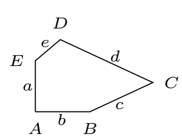
$$A = 90^\circ$$
$$2B + C = 360^\circ$$
$$C + E = 180^\circ$$
$$2a + c = d = e$$

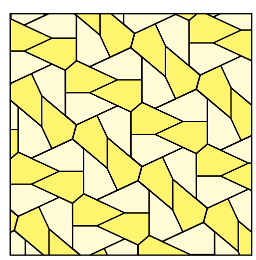




Type 12, 1976 Rice

$$A = 90^\circ$$
$$2B + C = 360^\circ$$
$$C + E = 180^\circ$$
$$2a = d = c + e$$

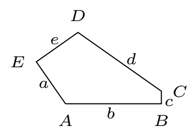


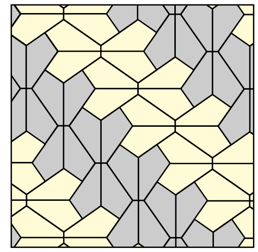


[5] Types 9 – 14 in [D. Schattschneider, Tiling the plane with congruent pentagons, *Math. Mag.* **51** (1978), 29-44]

Type 13, 1976 Rice

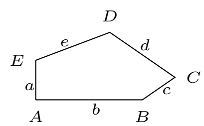
$$B = E = 90^\circ$$
$$2A + D = 360^\circ$$
$$2a = 2e = d$$

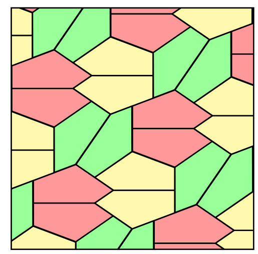




Type 14, 1985 Stein

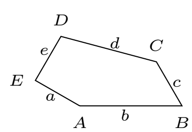
$$A = 90^\circ$$
$$2B + C = 360^\circ$$
$$C + E = 180^\circ$$
$$2a = 2c = d = e$$

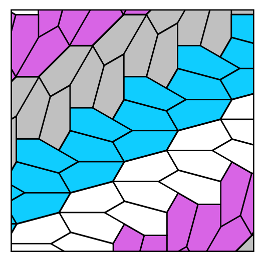




Type 15[6], 2015 Mann/McLoud/Von Derau

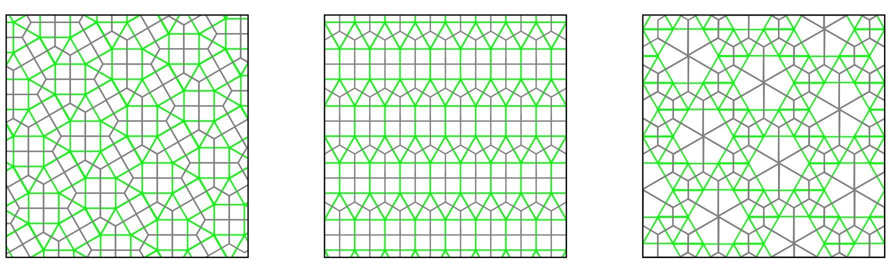
$$A = 150^\circ, B = 60^\circ$$
$$C = 135^\circ, D = 105^\circ$$
$$E = 90^\circ$$
$$a = c = e$$
$$b = 2a$$





Finding pentagonal tilings intuitively and systematically

We now want to have a look at how pentagonal tilings can be found. One way is through the Archimedean tilings we have seen earlier. Taking the duals (take the center of each polygon as the new vertex and join the vertices of adjacent polygons) of the three Archimedean tilings whose vertices are of valence 5, number 5, number 7 and number 8, gives us a pentagonal tiling, as seen below.



(a) Cairo tiling (b) House tiling (c) Floret tiling

Figure 6: Archimedean tilings and their duals

[6]A. Bellos, Attack on the pentagon results in discovery of new mathematical tile, *The Guardian* (11 Aug 2015)

But the Cairo tiling can also be discovered in other ways. By layering two hexagonal tilings of which one is at right angles to the other, the grid we get is the Cairo tiling. We can also find the Cairo tiling by dissecting a square into four congruent quadrilaterals and then joining the dissected squares together. Another way to discover pentagonal tilings from tilings by congruent convex pentagons is by dissecting each hexagon into two or more congruent pentagons. The House tiling, for example, can be found by experimenting – taking a tiling with squares and replacing one of the straight lines with a zig-zag line.

The first five types which Reinhardt found in 1918 are the only types that can generate a tile-transitive tiling. Tile-transitive means that its symmetry group acts transitively on the tiles. Kershner left out any assumption of tile transitivity. He only looked for tilings which were either edge-to-edge or in which every tile was surrounded by six vertices of the tiling. However, James found a tiling, Type 10, which was not edge-to-edge and consisted of tiles which are surrounded by 5 or 7 vertices. For this, he dissected a regular octagon into four congruent pentagons by perpendicular lines through its center. Marjorie Rice studied the corners of the tilings and analysed conditions on the angles. She found over forty different tilings of which one had not been discovered by then, Type 9. This showed that Kershner, though their searches were similar, was mistaken in eliminating the possibility of this type of edge-to-edge tiling.

For each pentagon there exists a tiling containing a minimal block of congruent pentagons. This minimal block has the property that the tiling consists of congruent images of this block and it can be mapped onto another congruent block by an isometry of the tiling. We say a tiling is n-block transitive when the minimal such block contains n pentagons. A tile-transitive tiling is hence a 1-block transitive tiling. Kershner's tilings are 2-block transitive tilings and we observe that each block is surrounded by six other blocks. Hence, these blocks are topological hexagons. We can see again that bisecting hexagons in hexagonal tilings can lead to pentagonal tilings. James' tiling is 3-block transitive and is also a topological hexagon. However, the pentagonal tiling of Type 9 by Rice is different. It is also 2-block transitive but the blocks are topological quadrilaterals since each block is surrounded by 4 blocks. This is rather surprising since the bisection of a quadrilateral cannot result in pentagons. Rice then observed that some of these 2-block transitive tilings can also be seen as 4-block transitive tilings which have the outline of two hexagons attached to each other. This is how Rice eventually discovered her other three tilings (Types 11-13). Type 14 is also 3-block transitive and a topological hexagon and so is type 15. It is therefore assumed that a pentagon tiles the plane only if there exists an n-block transitive tiling by that pentagon for $n \leqslant 3$.

Tiling with equilateral convex pentagons

In 1983, M. D. Hirschhorn and D. C. Hunt proved the following theorem[7]:

Theorem 2. *An equilateral convex pentagon tiles the plane if and only if it has two angles adding to* $180°$*, or it is the unique equilateral convex pentagon* X *with angles* A, B, C, D, E *satisfying*

- $A + 2B = 360°$
- $C + 2E = 360°$
- $A + C + 2D = 360°$.

This implies that $A \approx 70.88°, B \approx 144.56°, C \approx 89.26°, D \approx 99.93°$ *and* $E \approx 135.37°$.

We can see that Type 1, 2, 4, 7 and 8 can be equilateral, where Type 1, 2, 4 and 8 have two angles adding up to 180° and Type 7 is the unique pentagon X.

PROOF OUTLINE. Only consider edge-to-edge tilings, as non-edge-to-edge tilings can be transformed into edge-to-edge tilings.

Since we want the angles to match up such that there are no overlaps or gaps, we need the angles to have the relation

$$\alpha_A A + \alpha_B B + \alpha_C C + \alpha_D D + \alpha_E E = 360° \tag{1}$$

with $\alpha_A, \alpha_B, \alpha_C, \alpha_D, \alpha_E \in \mathbb{N}^+$.
To ensure that our polygon is a convex pentagon, we need the condition that

$$180° > A, B, C, D, E > \cos^{-1}\left(\frac{7}{8}\right) \quad \left(\text{and} \cos^{-1}\left(\frac{7}{8}\right) > 28°\right) .$$

We therefore get

$$360° = \alpha_A A + \alpha_B B + \alpha_C C + \alpha_D D + \alpha_E E > 28° \sum_{i=1}^{E} \alpha_i$$

and so

$$12 \geqslant \sum_{i=1}^{E} \alpha_i \, .$$

Similarly,

$$360° = \alpha_A A + \alpha_B B + \alpha_C C + \alpha_D D + \alpha_E E < 180° \sum_{i=1}^{E} \alpha_i$$

[7] M. D. Hirschhorn and D. C. Hunt, Equilateral convex pentagons which tile the plane, *Journal of Combinatorial Theory, Series* **A 39** (1983), 1–18

and so

$$3 < \sum_{i=1}^{E} \alpha_i \,.$$

This shows that there is only a finite set of relations.
Assume from now on that B is the largest angle and that $A \leqslant C$ to avoid duplications in the relations. Then, prove that

$$A \leqslant C \leqslant D \leqslant E \leqslant B \,. \tag{2}$$

Further analysis of the geometry of an equilateral convex pentagon gives

$$\begin{aligned} 108^\circ \leqslant B < 180^\circ \\ 180^\circ - \frac{1}{2}B - \sin^{-1}\left(\sin\left(\frac{1}{2}B\right) - \frac{1}{2}\right) \geqslant A \geqslant 180^\circ - B + 2\sin^{-1}\left(\frac{1}{4}\sin\left(\frac{1}{2}B\right)\right) \\ D = \cos^{-1}\left(\cos A + \cos B - \cos\left(A+B\right) - \frac{1}{2}\right) \\ C = 270^\circ - B - \frac{1}{2}D + \theta \\ E = 270^\circ - A - \frac{1}{2}D - \theta \end{aligned} \tag{3}$$

with

$$\theta = \tan^{-1}\left(\frac{\sin A - \sin B}{1 - \cos A - \cos B}\right).$$

We observe that the angles D, C and E are determined by the angles A and B and hence, an equilateral convex pentagon can be identified by a point in the AB-plane. Further reductions,

$$\begin{aligned} \cos^{-1}\left(\frac{7}{8}\right) < A \leqslant 108^\circ \\ 108^\circ \leqslant B < 180^\circ \\ 60^\circ < C \leqslant 108^\circ \\ \cos^{-1}\left(\frac{1}{4}\right) < D < 120^\circ \\ 108^\circ \leqslant E < 180^\circ \,, \end{aligned} \tag{4}$$

lead to only 220 solutions to (**??**). Of these, 13 do not actually equal 360°, 107 do not comply with (**??**), 6 are of Type 1 and 4 are of Type 2. This leaves 90 remaining relations, of which 14 involve B. In 7 of these 14, we have $\alpha_A > \alpha_B$, so they can be eliminated. We find that there are 54 pentagons which satisfy at least two relations including one of the 7 that involve B. Only 3 of these sets of relations involve all 5 angles and only one of them tessellates.

The tiling in this figure is a non-periodic tiling with an equilateral pentagon which has two adjacent angles adding up to 180°. It was discovered by Hirschhorn, published together with the above theorem, and is therefore known as the Hirschhorn tiling.

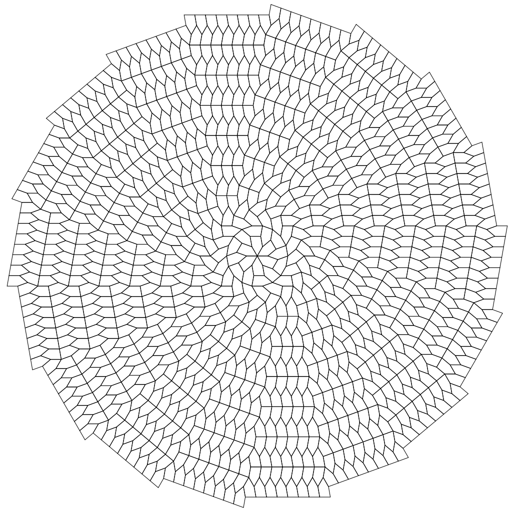


Figure 7:
The Hirschhorn tiling - a nonperiodic tiling with an equilateral convex pentagon

Tiling with convex pentagons with four equal-length edges

In 2005 and 2009, Sugimoto and Ogawa studied plane tilings with convex pentagons which have four equal-length edges ($a = b = c = d$) and presented a perfect list of such pentagons that tessellate. Again, we are just looking at edge-to-edge tilings[8,9]

1. $A + B + E = 360^\circ,\ B = E$
2. $B + D + E = 360^\circ$
3. $A + B + C = 360^\circ$
4. $2C + B = 2E + A = 360^\circ$
5. $C + D + E = 360^\circ,\ C = 2B$
6. $2E + C = 2B + A = 360^\circ$
7. $2D + B = 2E + A = 360^\circ$
8. $A + B + C = 360^\circ,\ C = 2D$
9. $A + B + C = 360^\circ,\ A = C$
10. $A + B + C = 360^\circ$
11. $C + D + E = 360^\circ$
12. $B + D + E = 360^\circ$
13. $C + D + E = 360^\circ,\ C = 2A$
14. $B + D + E = 360^\circ,\ B = 2A$

Summary

We saw that there exist eight semi-regular tessellations of the plane. When it comes to monohedral tiling, it was easy to see that all triangles and all quadrilaterals tessellate as well as three types of hexagons, including the regular hexagon. We listed all 15 types of convex pentagons that have been found to tile the plane and saw that some of them can be found intuitively, i.e. as duals of some semi-regular tilings, and systematically, i.e. with analysing tile-transitivity or 2-block/3-block-transitivity. For an equilateral

[8] T. Sugimoto and T. Ogawa, Systematic study of convex pentagonal tilings. I: Case of convex pentagons with four equal-length edges, *Forma* **20** (2005), 1–18

[9] T. Sugimoto and T. Ogawa, Errata: Systematic study of convex pentagonal tilings, II: tilings by convex pentagons with four equal-length edges, *Forma* **25** (2010), 49

convex pentagon to tile the plane, it has to have two angles adding to $180°$, or it is the unique equilateral convex pentagon with $A + 2B = 360°$, $C + 2E = 360°$ and $A + C + 2D = 360°$. Five of our 15 types can potentially be equilateral. Finally, we saw a complete list of pentagons with four equal-length edges which tile the plane.

Feeling inspired?

Richard James, a computer scientist, and Marjorie Rice, with no more than high school maths skills, demonstrate that anyone can contribute to solving the only remaining problem of monohedral polygonal tiling of the plane. So, becoming famous may never be easier – simply find a new pentagonal tiling or prove that there are not more than 15.