PROBLEM SECTION

Students are invited to submit solutions to one or more of these problems. Answers should bear the author's name, class, and school. Model solutions and the names of successful solvers will be published in the next issue of Parabola.

Only those students who have not yet passed the Intermediate Examination are eligible to submit solutions to problems in the Junior section. All students may submit solutions to problems in the Open Section.

JUNIOR

The first three of these problems were set in the South Australian School Mathematics Competition, May, 1964.

J1. (a) Two numbers, a and b are such that a is smaller and b is greater than 1. If S is the sum of a and b, and P is their product, prove that S and P differ by more than 1.

(b) Hence show that if the product of two positive numbers is 1, their sum cannot be less than 2.

(c) Using this result, or otherwise, prove that amongst all right-angled triangles of equal area, the isosceles triangle has the shortest hypotenuse.

J2. ABC is an equilateral triangle and P is a point on the circumference of its circumcircle. Prove that one of the distances PA, PB and PC is the sum of the other two.

J3. ABC is an equilateral triangle and P a point inside the triangle. Perpendiculars are drawn from P to the sides, the feet of the perpendiculars being X, Y, Z respectively. Prove that PX + PY + PZ is constant for all positions of P.

Show that this theorem can be extended to any convex polygon, (i. e. no interior angle greater than 180^{o}), with equal sides. (Some perpendiculars may meet the sides produced).

Is the converse of the theorem also true, i. e. if the sum of the lengths of the perpendiculars to the sides of a convex polygon is constant, are the sides of the polygon necessarily equal?

If the converse theorem is true, prove it; if it is not, give an example to show why it is false.

J4. Replace the letters by digits to make the following addition correct

AHAHA + TEHE = TEHAW.

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SMALL CHANGE

I have two coins in my pocket totalling 1/6d. One is not 6d. What are the coins?

(Solution page 16).

OPEN

The first three questions were set in the S. A. School Mathematics Competition, May, 1964.

O5. A mountain is of a perfectly conical shape. The base is a circle of radius 2 miles, and the steepest slopes leading up to the top are 3 miles long. From a point A at the southernmost point of the base a path leads up on the side of the mountain to B, a point on the northern slope and $\frac{2}{5}$ of the way to the top, (i. e. if T is the top and CT the northernmost slope, C being on the ground, then $BC = \frac{2}{5}$ CT).

If AB is the shortest path leading along the mountainside from A to B, find

(i) the length of the whole path AB,

(ii) The length of the part of the path between P and B where P is a point on the path where it is horizontal.

O6. The symbol x means the <u>integer part of</u> x, i. e. [x] is the <u>largest integer</u>, positive, negative or zero, which is <u>not more</u> than x. (E.g. [5] = 5, [7.3] = 7, [.5] = 0, [-1.6] = -2).

The numbers n and k are positive integers, and k is greater than 1.

(i) Prove that $\left[\frac{2n}{k}\right] \geq \left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right]$

(ii) Prove that for any number x

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx].$$

O7. A, B, C, D, E and F are six points. Straight line segments are drawn to connect each pair of the six points some in red and some in black ink. Prove that no matter how the colouring is done, there will always be three of the six points joined by lines of the same colour.

O8. The sides of a quadrilateral ABCD are of fixed length, but the angles may be varied. Prove that its area is largest when it is cyclic by using the isoperimetric property of the circle.

O9. Find the smallest integer whose value is trebled if the left hand digit is transferred to the right hand end.

O10. One morning after church the verger, pointing to three departing parishioners, asked the bishop, "How old are those three people?"

The bishop replied, "The product of their ages is 2,450, and the sum of their ages is twice your age."

The verger thought for some moments and said, "I'm afraid I still don't know."

The bishop answered, "I am older than any of them."

"Aha!" said the verger, "Now I know."

Problem: How old was the bishop? (Ages are in whole numbers of years and no one is over 100).

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