THE FOUR COLOUR PROBLEM

Whilst very few people doubt that there are exciting new discoveries awaiting the inspired researcher in Physics, or Chemistry, or almost any other science, many seem to imagine that virtually all mathematics has been known for fifty years or more. This is perhaps partly because of the difficulty of explaining briefly to the layman the nature and importance of many of the rather abstruse achievements of twentieth century creative mathematicians; whereas some of the significance of, say, the discovery of a new antibiotic by a biochemist can be grasped by anyone.

Yet large numbers of easily comprehended unsolved problems abound in all branches of mathematics. Some of these problems because of their fundamental nature have attracted the attention of mathematicians for many years. One which after a century remains a challenge to pure mathematicians, working in the branch of mathematics called topology, is known as the four colour problem.

Makers of atlases colour adjacent countries on maps with different colours. Countries are regarded as adjacent if they have a positive length of boundary in common, but not if they touch only at isolated points. Only maps with a finite number of countries are considered. What is the minimum number of colours required to achieve such a colouring for a given map? The map of Australia might be coloured using eight different colours - one for each state and blue for the sea; but no more than four are really necessary. In fact, no one has yet succeeded in drawing a map on a plane, or on a sphere, requiring more than four colours. On the other hand no one has been able to prove that four colours are sufficient for every map. An attempt to do so was published in 1879, but eleven years later the English mathematician Heawood found an error in the reasoning employed. By revising the argument he was able to show that five colours are always sufficient. The problem remains of either improving Heawood's result to four colours, or of producing a map requiring five colours.

One might suppose that since the problem has proved so frustrating for maps on a plane or a sphere, it would be quite unmanageable for maps drawn on a more complicated surface, such as the surface of a doughnut (a torus, to give it its correct name). But strangely enough, it is completely solved for this surface. We shall show later that no map on the torus requires more than seven colours. Fig 1 shows how to obtain a map for which all seven are required. First fold the rectangle into a cylinder by bringing the edge A_1A_2 into coincidence with A_3A_4, then imagine this cylinder bent round into a complete circle so that its (circular) end A_1A_3 coincides with A_2A_4. Points labelled with the same letter will now coincide. Regions bearing the same number will form one country on the resulting torus, and we obtain a map of seven countries each of which is adjacent to the other six.

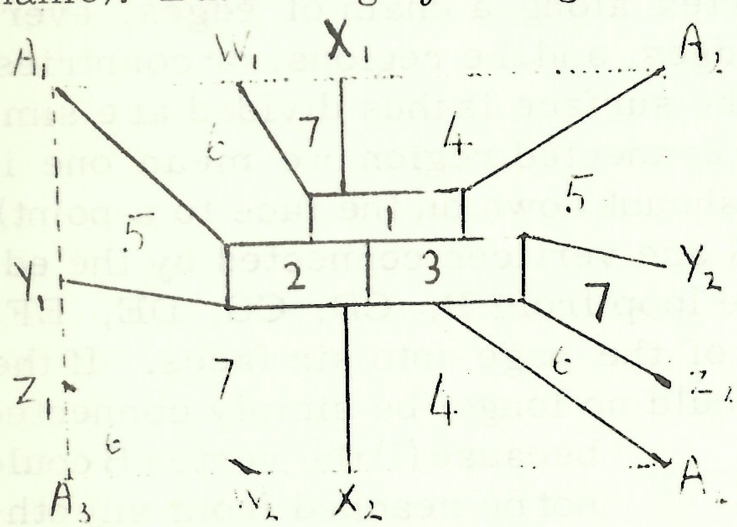

Fig. 1

Before outlining a possible method of proving the results quoted, we dispose of the simple fact that the map colouring problems on the sphere and plane are equivalent. For, given any map on a sphere, one may imagine a small hole made in the interior of one country, and the resulting punctured spherical surface continuously deformed into a plane map (allowing stretching but not tearing of the surface). Conversely any plane map may, by reversing the process, be represented on a sphere. (The additional point needed to complete the sphere can at most introduce one-point contact of non-adjacent countries). We need henceforth consider only maps on a sphere.

Let us now analyze the structure of maps on any surface. Any "simply connected map" may be constructed in the following way. A finite number of points, to be called vertices, are selected on the surface. Non-intersecting lines, to be called edges, are drawn linking these vertices in such a way that any vertex can be reached from any other vertex along a chain of edges, every vertex terminates at least two edges, and the regions, or countries, to be called faces, into which the surface is thus divided are simply connected. (By a simply connected region we mean one in which any closed curve can be shrunk down on the face to a point). In Fig 2, A, B, C, D, E, F, G are vertices connected by the edges AB, AC, AD, AE, AF, the loop from B, CD, CG, DE, EF, FG, which divide the surface of the page into six faces. If the edge AB were absent the map would no longer be simply connected because (i) the vertex B could not be reached from any other vertex, and (ii) a closed curve could be drawn in face 2, encircling face 1, which could not be shrunk down to a point in 2.

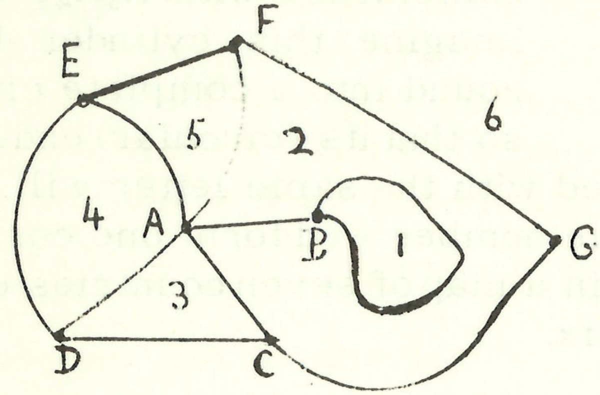

Fig. 2

If the edge FG were absent, the vertex G would terminate less than two edges and the figure would not be a map under our definition. Note that only two edge end-points coincide at the vertex G, whilst three or more coincide at all the other vertices. Because of this property, G could be omitted and the edges FGC united to form a single edge FC without essentially altering the map as far as colouring is concerned.

An essential step in the proof of Heawood's five colour theorem is Euler's Formula. This asserts that for any simply connected map on a sphere, $F + V - E = 2$, where F is the number of faces, V the number of vertices, and E the number of edges. Euler first proved the formula for simple polyhedra - solids all of whose faces are plane polygons, for example a cube or a tetrahedron. If the edges of such a solid are replaced by ink lines and its surface deformed into a sphere, a map of the type we are considering results.

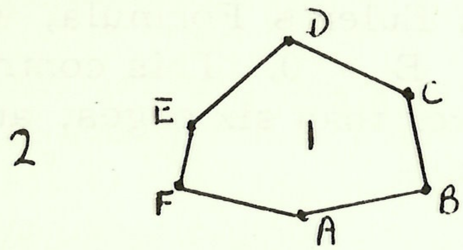

Fig. 3

Euler's Formula is obviously true for a map on a sphere with only two faces, since if the boundary separating them contains n vertices it also has n edges (Fig 3). Mow suppose we are given a simply connected map on a sphere with F (>2) faces. Rubbing out an edge which separates two faces, (and such an edge must exist) will reduce both the number of edges and the number of faces by one, leaving a network with $F + V - E$ unchanged. However, such an operation may have left a vertex which terminates only one edge, and both vertex and edge must be deleted before another simply connected map can be obtained. Proceeding in this fashion we eventually reach the situation of Fig 3 and since $F + V - E$ was unaltered at each stage, its value must initially have been 2.

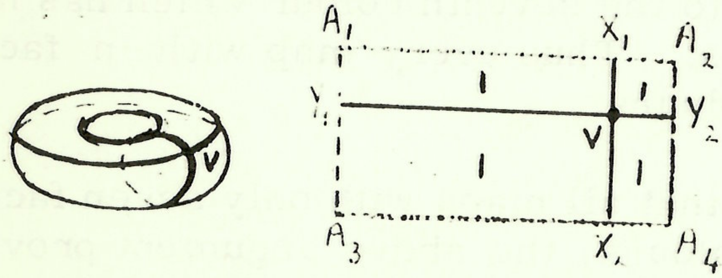

Fig. 4

For maps drawn on a torus the appropriate formula is $F + V - E = 0$. Fig 4 shows a simply connected map on a torus with one vertex, two edges, and one face.

Using Euler's Formula we now show that for any simply connected map on a sphere there is at least one face having five or fewer neighbours. For a map with no more than six faces this statement is trivially true. A map with more than six faces will be essentially unchanged if all vertices where only two edge end-points coincide are deleted (e.g. G in Fig. 2). Since each edge has two end-points and now at each vertex three or more end-points coincide, $2E \geqslant 3V$. Let us suppose that every face has six or more edges. Then 6F is less than or equal to the number obtained on counting the edges face by face. Since in this process no edge is counted more than twice, it follows that $2E \geqslant 6F$. Substituting $V \leqslant 2E/3$ and $F \leqslant E/3$ in Euler's Formula, we obtain $2 = F + V - E \leqslant E/3 + 2E/3 - E = 0$. This contradiction shows that at least one face has fewer than six edges, and hence no more than five neighbouring faces.

Exactly the same argument shows that for maps on the torus there is always a face with six or fewer neighbours. Using this we can now show that every map on the torus can be coloured with seven colours. Assume that every map with fewer than n faces can be coloured with seven colours, and consider any map with n faces. One of these has no more than six edges. Delete one of these (AB say), to obtain a map with n - 1 faces. Colour this with seven colours in accordance with the assumption, on six at most being needed for the section shown in Fig. 5. On restoring the edge AB, the colour of the original face may now be changed to the seventh colour which has not been used on its six neighbours. Thus every map with n faces may be coloured with only 7 colours.

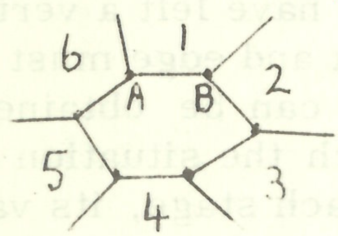

Fig. 5

Since it is obvious that all maps with only seven faces can be coloured with seven colours, the above argument proves that the same is true of all maps with eight faces, then all with nine faces, in fact eventually all maps with any finite number of faces.

An identical argument would show that every map on a sphere could be coloured with six colours. This result can be improved to five colours in a number of ways. For example, assume that every map with fewer than n faces can be coloured with five colours, and consider any map with n faces. Find a face with no more than five neighbours. If it has only four (or fewer) neighbours we proceed as before, deleting one edge. However, if it has five neighbours, we can always find two which do not touch. For example, if as on Fig. 6, face 3 touches face 5, then face 4 certainly cannot touch face 2. Delete the boundaries AB and CD and colour the resulting map of n-2 faces with five colours. On restoring the boundaries the colour of the face ABDC may now be changed to complete the colouring of the original map, since faces 2 and 4 have the same colour, and there are at most four colours used on neighbouring faces.

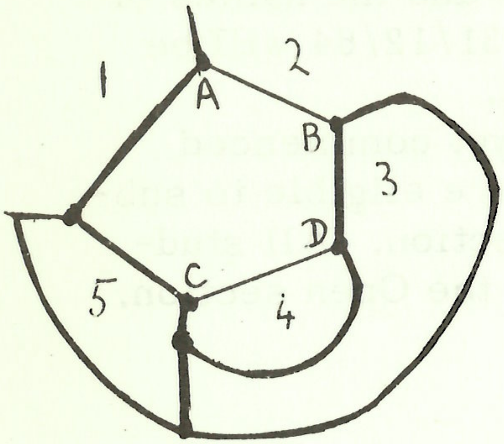

Fig. 6

You should not assume that topology is concerned solely with comparatively unimportant mathematical curiosities such as the four colour problem. It is still a rapidly growing branch of mathematics whose importance is recognised by both pure and applied mathematicians.

Reference:- R. Courant and H. Robbins, What is Mathematics, Oxford University Press, N. Y., (1951), Ch. V.

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