SOLUTIONS of the COMPETITION PROBLEMS.

Junior Q1. If a_1, a_2, and a_3, are positive numbers whose sum is two, prove that $a_1a_2 + a_2a_3 \leqq 1$.

Senior Q1. If a_1, a_2, a_3, and a_4 are positive numbers whose sum is A, show that $a_1a_2 + a_2a_3 + a_3a_4 \leqq A^2/4$.

We shall prove the general statement:- If a_1, a_2, a_3,...., and a_n are positive numbers whose sum is A, then

$$a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n \leqq A^2/4.$$

This result includes both questions.

Solution. If A and B are real numbers, $(A + B)(A - B) = A^2 - B^2 \leqq A^2$, since $B^2 \geqq 0$. Put $A = a_1 + a_2 + \ldots + a_n$, and $B = a_1 - a_2 + a_3 - \ldots \pm a_n$. Then $A + B = 2(a_1 + a_3 + \ldots)$ the sum of all a's with odd subscripts, and $A - B = 2(a_2 + a_4 + a_6 + \ldots)$, the sum of all a's with even subscripts.

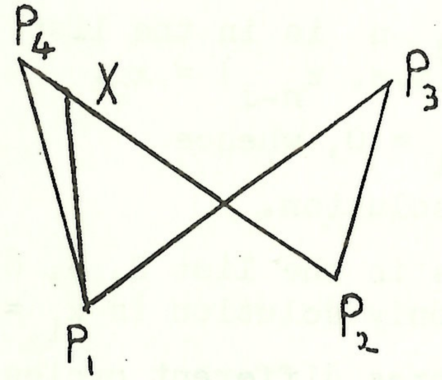
Hence $4(a_1 + a_3 + a_5 + \ldots)(a_2 + a_4 + a_6 + \ldots) \leqq A^2$.

Multiplying out the two expressions in parentheses yields all possible terms a_ia_j with one subscript odd and one even. Included amongst these are a_1a_2, a_2a_3, ..., $a_{n-1}a_n$, and, if $n > 3$, others as well (e.g. a_1a_4) all of which are positive. Omitting these other terms makes the L.H.S. even smaller; hence

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n \leqq (a_1 + a_3 + \ldots)(a_2 + a_4 + \ldots) \leqq A^2/4.$$

Junior Q2. Given any four points in the plane, no three of which lie on a straight line, prove that it is possible to choose three of them, P_1, P_2, and P_3 say, such that the circle through them either encloses, or goes through, the fourth point P_4.

Proof. Label the points in such a way that P_3 and P_4 lie



on the same side of the line P_1P_2 and $\angle P_1P_3P_2 \leq \angle P_1P_4P_2$. If these angles are equal, the points are concyclic, and there is nothing more to be proved. But if $\angle P_1P_3P_2 < \angle P_1P_4P_2$, then P_4 is an interior point of the circle $P_1P_2P_3$. For suppose P_4 lies outside the circle, and let P_2P_4 cut the circle at X. Then $\angle P_1P_3P_2 < \angle P_1P_4P_2 < \angle P_1XP_2$ (exterior angle of a triangle) $= \angle P_1P_3P_2$ (on the same arc). This contradiction shows that the supposition was false.

Junior Q3. The following are n simultaneous equations in the unknowns $x_1, x_2, \ldots\ldots x_n$.

$$\begin{aligned}
x_1 + x_2 + x_3 &= 0\\
x_2 + x_3 + x_4 &= 0\\
x_3 + x_4 + x_5 &= 0\\
&\cdots\cdots\\
&\cdots\cdots\\
x_{n-2} + x_{n-1} + x_n &= 0\\
x_{n-1} + x_n + x_1 &= 0\\
x_n + x_1 + x_2 &= 0.
\end{aligned}$$

For what values of n do these equations have a unique solution? When the solution is not unique, write down the most general solution.

<u>Answer</u>. Subtracting the first two equations gives $x_1 = x_4$. Similarly from the fourth and fifth, $x_4 = x_7$; and from the third last and the second last $x_{n-2} = x_1$. Hence:-

$$x_{n-2} = x_1 = x_4 = x_7 = x_{10} = \dots \quad \text{and similarly}$$

$$(1) \quad x_{n-1} = x_2 = x_5 = x_8 = x_{11} = \dots \quad \text{and}$$

$$x_n = x_3 = x_6 = x_9 = x_{12} = \dots$$

(a) If $n = 3k + 1$ where k is an integer, n is in the list 1, 4, 7, 10, ... and $x_1 = x_n = x_3 = x_{3k}$ (i.e. x_{n-1}) $= x_2$. Therefore $x_1 + x_2 + x_3 = 0$ becomes $3x_1 = 0$, whence $0 = x_1 = x_2 = x_3 = x_i$ $(i \leq n)$, the only solution.

(b) Similarly, if $n = 3k + 2$, (i.e. n is in the list 2, 5, 8,..) $x_2 = x_n = x_3 = x_{n-2} = x_1$ and again the only solution is $x_i = 0$.

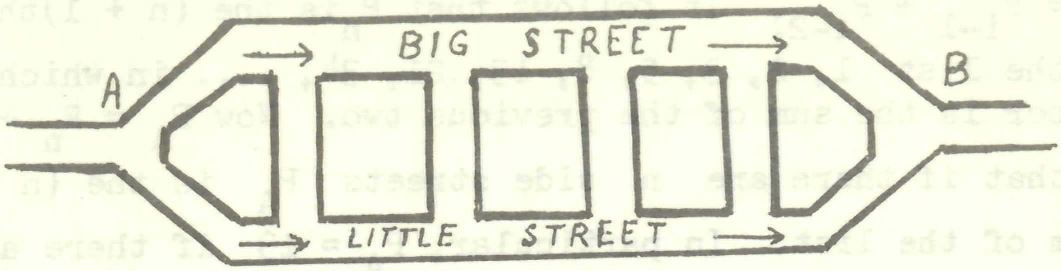
(c) If $n = 3k$, the equations (1) form three different cycles. All the given equations are satisfied in this case by

$$x_1 = x_4 = x_7 = \dots\dots = a$$

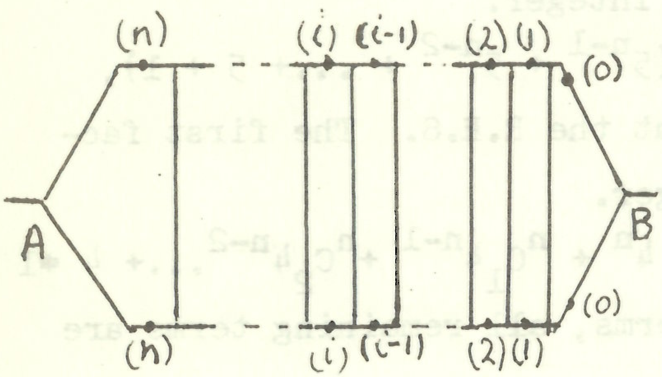
$$x_2 = x_5 = x_8 = \dots\dots = b$$

$$x_3 = x_6 = x_9 = \dots\dots = -a-b,$$ where a and b are any numbers.

<u>Junior Q4</u>. In the accompanying street map, Big St. and Little St. are both one-way (they carry East-bound traffic only). By how many different routes can one drive from A to B? How many routes are there if the number of North-South side streets is altered from four to n? If the side streets are made one-way, carrying North-bound and South-bound traffic alternately, how many routes go from A to B on the given map? How many routes are there for 10 side streets?



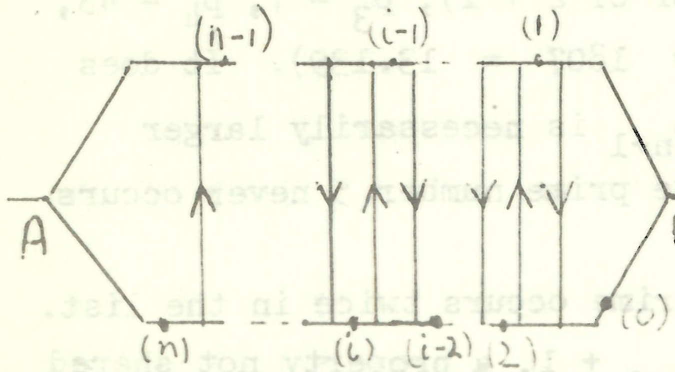
<u>Answer.</u> (i) All side streets 2-way.



By symmetry, there are the same number of routes from either of the points labelled (i), (i = 0, 1, 2, ...n) to B. Let this number be N_i. Then $N_0 = 1$, obviously. Also from a point (i) there are N_{i-1} routes if the first side street is taken, and a further N_{i-1} if it is not. Hence $N_i = 2N_{i-1}$.

Thus $N_1 = 2.1 = 2^1$; $N_2 = 2N_1 = 2^2$; and eventually $N_n = 2^n$. The number of routes from A to B is $N_n + N_n = 2^{n+1}$. For the given diagram, n = 4, and the number of routes is 2^5 (= 32).

(ii) Side streets one-way, carrying North-bound and South-bound traffic alternately.



Let P_i, (i = 0, 1, 2, ...n) and P_A be the number of different routes to B from the point (i) or from A. Obviously $P_0 = 1$ and $P_1 = 2$. From the point (i) there are P_{i-1} routes if the first side street is taken, and P_{i-2} if it is not. Hence

$P_i = P_{i-1} + P_{i-2}$. It follows that P_n is the $(n + 1)$th term of the list 1, 2, 3, 5, 8, 13, 21, 34, ... in which each number is the sum of the previous two. Now $P_A = P_n + P_{n-1}$, so that if there are n side streets P_A is the $(n + 2)$th term of the list. In particular, $P_A = 13$ if there are four side streets, and $P_A = 233$ if there are ten side streets.

Junior Q5 and Senior Q2. (i) Show that $5^n - 1$ is divisible by four, n being any positive integer.

Proof. (a) $5^n - 1 = (5 - 1)(5^{n-1} + 5^{n-2} + \ldots + 5 + 1)$, as may be seen by multiplying out the R.H.S. The first factor is four, the second an integer.

(b) $(4 + 1)^n - 1 = 4^n + {}^nC_1 4^{n-1} + {}^nC_2 4^{n-2} \ldots + 4 + 1 - 1$

After cancelling the last two terms, all remaining terms are integers divisible by 4.

There are other simple proofs.

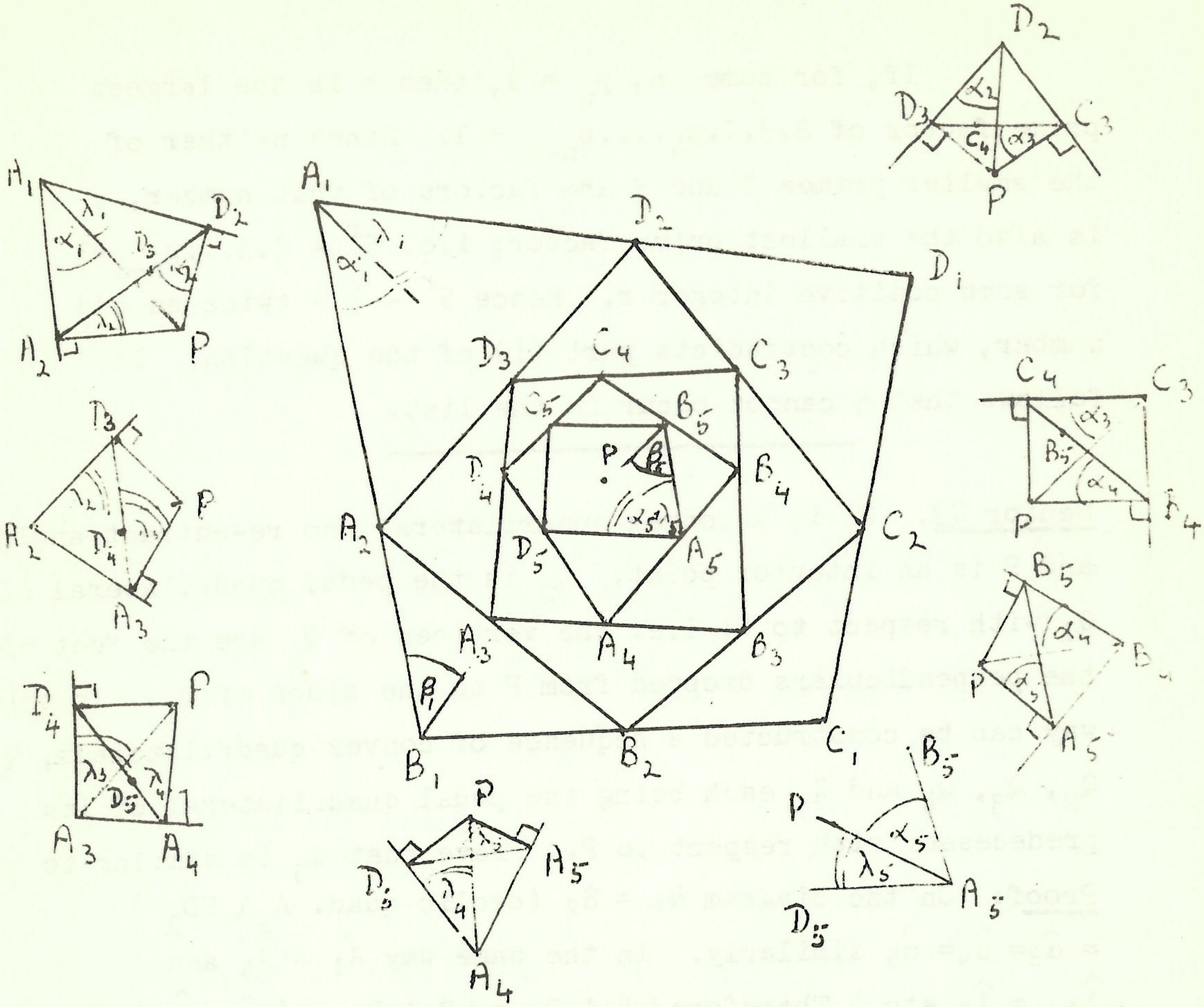
(ii) A list of prime numbers p_1, p_2, $p_3, \ldots\ldots p_n, \ldots$ is generated as follows: $p_1 = 2$, and if $n > 1$, p_n is the largest prime factor of $p_1 p_2 p_3 \ldots p_{n-1} + 1$. Thus $p_2 = 3$, (the largest prime factor of $2 + 1$), $p_3 = 7$, $p_4 = 43$, $p_5 = 139$, (since $2.3.7.43 + 1 = 1807 = 13.139$). It does not follow from this rule that p_{n+1} is necessarily larger than p_n. However, prove that the prime number 5 never occurs in the list.

Proof. First we show that no prime occurs twice in the list. For p_n is a factor of $p_1 p_2 \ldots p_{n-1} + 1$, a property not shared by p_1, p_2, ... or p_{n-1}, so that $p_n \neq p_i$, $i < n$. In particular, 2 occurs only once.

If, for some n, $p_n = 5$, then 5 is the largest prime factor of $2.3.7.p_4....p_{n-1} + 1$. Since neither of the smaller primes 2 and 3 are factors of this number, 5 is also the smallest prime factor; i.e. $5^r = 2.3...p_{n-1}+1$ for some positive integer r. Hence $5^r - 1$ = twice an odd number, which contradicts part (i) of the question. It follows that 5 cannot occur in the list.

Senior Q3. Q_1 is a convex quadrilateral (no re-entrant angle) and P is an interior point. Q_2 is the pedal quadrilateral of Q_1 with respect to P; i.e. the vertices of Q_2 are the feet of the perpendiculars dropped from P to the sides of Q_1. In this way can be constructed a sequence of convex quadrilaterals, Q_1, Q_2, Q_3, Q_4 and Q_5 each being the pedal quadrilateral of its predecessor with respect to P. Prove that Q_5 is similar to Q_1.

Proof: On the diagram $\hat{\alpha}_1 = \hat{\alpha}_2$ (cyclic quad. $A_1A_2PD_2$) $= \hat{\alpha}_3 = \hat{\alpha}_4 = \hat{\alpha}_5$ similarly. In the same way $\hat{\beta}_1 = \hat{\beta}_5$ and $\hat{\lambda}_1 = \hat{\lambda}_5$ etc. Therefore $\angle B_1A_1D_1 = \angle B_5A_5D_5 = (\hat{\alpha} + \hat{\lambda})$, and similar working using the other vertices shows that Q_1 and Q_5 are equiangular. This is not yet sufficient to ensure that Q_1 is similar to Q_5. (Consider a square and a rectangle). To prove the sides proportional:- The triangles PA_1B_1, PB_1C_1, PC_1D_1 ,and PD_1A_1 are by the above working equiangular with the triangles PA_5B_5 PB_5C_5, PC_5D_5 and PD_5A_5 respectively. Hence $A_1B_1/A_5B_5 = PB_1/PB_5 = B_1C_1/B_5C_5 = PC_1/PC_5 = C_1D_1/C_5D_5 = PD_1/PD_5 = D_1A_1/D_5A_5$.



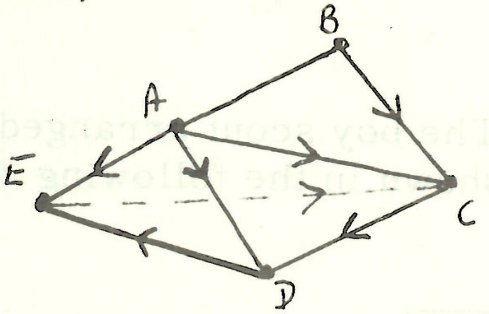
Senior Q4. Seven towns T_1, T_2, ...T_7 are connected by a network of 21 one-way roads such that exactly one road runs between any two towns. The authorities have succeeded in directing the roads in such a manner that given any pair of towns T_i and T_j, there is a third town T_k, such that T_k can be reached by a direct route from both T_i and T_j.

(i) Prove that of the 6 roads with an end at any town T_i, the number in which traffic is directed away from T_i is at least 3. Hence, prove that it is exactly 3.

(ii) Let the towns which can be reached directly from T_1 be numbered T_2, T_3, and T_4. Show that the roads between T_3, T_2, and T_4 form a circuit.

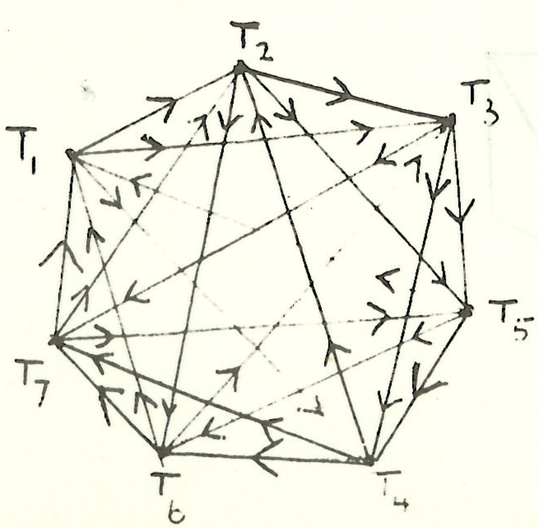
(iii)Display on a sketch a possible orientation of traffic in the 21 roads.

<u>Answer.</u> (i)Let A be any one of the towns. If B is any other there is a third town C which can be reached directly from both A and B. Hence at least one road leaves A. There is a town D which can be reached directly from both A and C. Now consider A and D. There is a town E (possibly the same town as B) which can be reached directly from both A and D. E is not the same town as C since in the road CD traffic is directed from C to D, whilst in the road DE traffic runs from D to E. Hence there are at least three towns C,D, and E which can be reached directly from A. Since there are only 21 roads and we have shown that at least 3 leave each of the 7 towns, there must be exactly 3 leaving (and therefore 3 entering) each town.



(ii) With the same notation as in (i) traffic is known to flow from C to D and from D to E. We have only to show that in the road EC traffic flows from E to C. Consider A and E. There is a town, X, which can be reached directly from each. The only towns apart from E which can be reached directly from A are C and D. But X cannot be D since traffic moves from D to E. Hence X is C and traffic moves from E to C.

(iii)



CAN YOU SEE THIS?

FRONT and TOP BACK and BOTTOM

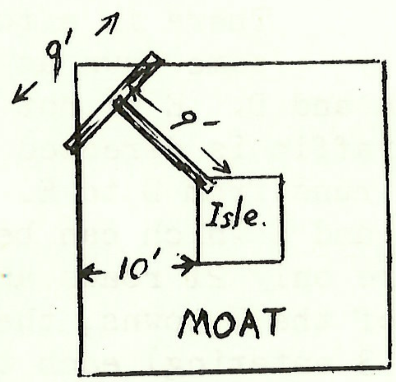
This object has plane faces and has been cut from a block of wood.

Draw a three dimensional sketch of such an object.

(Answer Page 28).

SOLUTIONS.

PREPARED (p 12). The boy scout arranged the planks at a corner, as shown in the following diagram.



BRAINTEASER (p 14). The wise man said, "let each ride the other's horse".

CAN YOU SEE THIS ? (p. 27).