PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class, and school. Model solutions and the names of those who send correct solutions by 30/4/65 will be published in the next issue of PARABOLA.

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior section. All students may submit solutions of problems in the Open section.

JUNIOR

The first two problems were set in the Queensland School Mathematics Competition, 1964.

J21. In a box there is a number of balls of c different colours. What is the smallest number of balls for which we can say that, however the colours are distributed, there is at least one set of s balls of the same colour? Justify your statement.

Further, if $n < c$, what is the smallest number of balls for which we can say that there are at least n sets, each set containing s balls of the one colour? (In forming these sets we do not require that sets be of different colours; e.g. from $2s$ balls coloured red we could form 2 sets.)

J22. There are seven points on a straight line in the order A, B, C, D, E, F, G, and P is a variable point on the line. For what position of P is the sum of the positive lengths

$$PA + PB + PC + PD + PE + PF + PG$$

least? Justify your statement. (continued on p. 10)

J22. (cont.) Answer a similar question for six given points U, V, W, X, Y, Z on a line, and a variable point Q.

J23. (a) Prove that, if n is a positive integer greater than 2, the numbers $2^n + 1$ and $2^n - 1$ are not both prime.

(b) Prove that of the three positive integers a, 8a - 1 and 8a + 1, there is at least one which is not prime.

J24. All the odd integers, beginning with 1, are written successively (that is, 1 3 5 7 9 11 13 15 17 19 21 ...). Which digit occupies the 100,000th position?

OPEN

The first three problems were set in the Queensland School Mathematics Competition, 1964.

O25. If S_r denotes the sum of the rth powers of the numbers 1, 2, 3, ..., 9, show by considering the sign of

$$(1 + x)^2 + (2 + 4x)^2 + (3 + 9x)^2 + \dots + (9 + 81x)^2$$

that $S_3^2 < S_2 S_4$.

Give an outline proof also of the following: -

$$S_2^2 < 9S_4 ;$$

$$S_{m+n}^2 < S_{2m} S_{2n}, \text{ if } n \neq m.$$

State and justify - in a few lines of discussion - more general results applicable to sets of numbers other than 1, 2, ..., 9.

HORSE SENSE. Put 20 horses in 5 stalls so that there is an odd number of horses in each stall. (Answer p. 20)

O26. We say that a plane is transformed by a similitude, of centre O (on the plane) and ratio k, if each point P on the plane is transformed into a point P' on the line OP, such that

$$OP' = k.OP.$$

If k is negative, the senses of the segments OP and OP' are opposite.

ABC is a triangle whose centroid is G. Show that, under a similitude of centre G and ratio $k = -\frac{1}{2}$, the points A, B, C transform into A', B', C', the mid-points of the sides. Hence show that H, G, and S are collinear, H being the orthocentre and S the circumcentre of the triangle ABC. If, on the line HGS, G is taken as origin and S has co-ordinate -1, what is the co-ordinate of H?

If the lines drawn parallel to BC, CA, AB through A, B, C respectively meet in A_1, B_1, C_1, show that H_1, the orthocentre of the triangle $A_1B_1C_1$, lies on HGS. What is its co-ordinate in the above system?

O27. Consider all points in the co-ordinate plane XOY whose co-ordinates are both positive integers. The sum $x + y$ of the co-ordinates of such a point is called the index of the point. I write down the point (1, 1) of index 2 ; then the points (1, 2), (2, 1) of index 3 in ascending order of their first co-ordinates; then the points (1, 3), (2, 2), (3, 1) of index 4 in ascending order of their first co-ordinates; and so on. The result is ;- (1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), (2, 3), (3, 2), (4, 1), (1, 5), ...

If (x, y) is the nth point written down, find a formula for n in terms of x and y.

RISE OR FALL. A fisherman rows out to the middle of a lake and throws overboard some pieces of cast iron. What happens to the level of the water in the lake? (Answer p. 20)

O28. (a) $\angle AOB$ is an acute angle and OX is a variable ray dividing it into two smaller angles. Show that

$$\sin \angle AOX + \sin \angle BOX$$

is greatest when OX bisects $\angle AOB$.

(b) A polygon with n sides is inscribed in a circle. Show that the area of the polygon is greatest if it is regular.

O29. Two people, A and B play a game in which they alternately remove at least one and not more than three matches from a heap. A makes the first move. The player who is forced to take the last match loses. If there are initially 60 matches in the heap, which player should win? What is his correct strategy?

O30. In a certain garden there are a number of beds each planted with flowers. If x and y are two different varieties of plants growing, there is exactly one bed containing both. If any bed is considered there is one and only one other bed in which none of the same varieties of plants are growing.

Show that (i) every bed has at least 2 varieties of plants

(ii) there are at least 4 varieties of plants in the garden;

(iii) there are at least 6 flower beds;

(iv) no bed has more than 2 varieties of plants.

SIMPLE ARITHMETIC

In the accompanying long division, all save one of the digits have been obliterated. Can you replace them?

(Answer p. 20)

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                    x x 8 x x
        x x x ) x x x x x x x x
                x x x
                  x x x x
                    x x x
                      x x x x
                      x x x x
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