SOLUTIONS OF PROBLEMS IN PARABOLA, VOL. 1, NO. 2.

J11. A drill squad contains mn people, arranged in m rows of n people. In each of the n columns so formed, the lightest person is noted, and the heaviest of these is found to be Smith. Then the heaviest person in each row is noted, and the lightest of these is found to be Jones. Of Smith and Jones, who is the heavier?

Answer. Jones is heavier than Smith. If they are in the same column this is obvious, since Smith is the lightest in his column. If they are in the same row it is again obvious, since Jones is the heaviest in his row. Otherwise, find the person (Robinson, say) who is in the same column as Smith and in the same row as Jones. Then Jones is heavier than Robinson, who is in turn heavier than Smith; hence the stated result.

(Correct solution from J. Taylor, J. J. Cahill M. H. S. P. Treloar, Toronto H. S., submitted the correct answer, but without working.)

J12. (i) Show that 120 is a divisor of $n^5 - 5n^3 + 4n$ for every integer n.

(ii) Show that 49 is not a divisor of $n^2 + n + 2$ for any integer n.

Answer. (i) $n^5 - 5n^3 + 4n = n.(n^4 - 5n^2 + 4) = n(n^2 - 4)(n^2 - 1)$

$= (n - 2)(n - 1)n(n + 1)(n + 2)$, the product of five consecutive integers. One of these must be divisible by 5, at least one divisible by four, and at least one divisible by three. Since at least two of the numbers are even, there is at least one of them which is divisible by two in addition to that divisible by four. Hence the product is divisible by 5 x 4 x 3 x 2 = 120. (Note that 0 is divisible by 120; indeed, it is divisible by every number.)

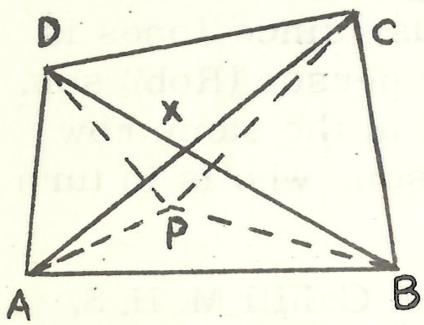
(Correct solution from J. Taylor, J. J. Cahill M. H. S.; partly correct solution from T. D. Potts, Newington College).

(ii) $n^2 + n + 2 = (n + 4)^2 - 7(n + 2)$

If $n^2 + n + 2$ is divisible by 49, it is certainly divisible by 7, and therefore $(n + 4)^2$ is divisible by 7. But in this case, 7 is a factor of $(n + 4)$, and therefore not of $(n + 2)$. Of the terms on the R. H. S. of the above equation, the first is divisible by 49, but not the second, so that their difference is not.

(Correct solution from J. Taylor, J. J. Cahill M. H. S.)

J13. ABCD is a convex quadrilateral and P is a variable point. For what position of P is the sum of the lengths PA + PB + PC + PD least?



Answer. P should coincide with X, the point of intersection of the diagonals AC and BD. Then PA + PB + PC + PD = AC + BD. Indeed, if P is any point, PA + PC $\geqslant$ AC, with equality only if P is on AC, and PB + PD $\geqslant$ BD, with equality only if P is on BD. Adding, we obtain PA + PB + PC + PD $\geqslant$ AC + BD with equality only if P coincides with X.

(Correct solutions from J. Taylor, T. D. Potts)

J14. In the following calculation, consisting of a long multiplication of two 2-digit numbers, followed by the addition of a 3-digit number, all save one of the digits have been obliterated. Find what they must have been.

```
             x x
multiply     x x
           x x x
         x x x
         x x x x
add        1 x x
       x x x x x
```

Answer. The final answer, having five digits, is not less than 10,000. The third last line must be a number not less than 9801, since the addition of a number between 100 and 200 gives at least 10000. But, since it is also the product of two whole numbers less than 100, it cannot be greater than 9801 (= 99 x 99). It must therefore be equal to 9801 and the only way to complete thecalculation is as shown below.

$$\begin{array}{lr} & 99 \\ \text{multiply} & \underline{99} \\ & 891 \\ & \underline{891} \\ & 9801 \\ \text{add} & \underline{199} \\ & \underline{10000} \end{array}$$

(Correct solutions from J. Taylor (J. J. Cahill H. S.); R. Holliday (Toronto H. S.); D. Hermann (Oak Flats H. S.); J. Vogt (De La Salle College, Ashfield); and P. Treloar (Toronto H. S.)).

O15. Find all solutions of the following simultaneous equations:

$$xy + wz = 20$$
$$xz + wy = 20$$
$$xw + yz = 20$$
$$x + y + z + w = 14.$$

Answer. $(x + y - z - w)^2 = (x + y + z + w)^2 - 4(xz + wy) - 4(xw + yz)$

$$= 14^2 - 4\times20 - 4\times20 = 36.$$

Therefore	$x + y - z - w = \pm 6.$	(1)
Similarly	$x - y - z + w = \pm 6$	(2)
and	$x - y + z - w = \pm 6$	(3)
Also	$x + y + z + w = 14$	(4)

There are eight different ways of choosing the signs in equations (1), (2), and (3), and these yield eight different solutions. For example, if all signs are taken positive, we obtain:-

(1) + (2) + (3) + (4)	$4x = 32;\ x = 8.$
(1) + (4) - (2) - (3)	$4y = 8;\ y = 2.$
(3) + (4) - (1) - (2)	$4z = 8;\ z = 2.$
(2) + (4) - (1) - (3)	$4w = 8;\ w = 2.$

Thus one solution is (w, x, y, z) = (2, 8, 2, 2). The other solutions may be similarly obtained, and are (8, 2, 2, 2) ; (2, 2, 8, 2) ; (2, 2, 2, 8) ; (5, 5, 5, -1) ; (5, 5, -1, 5) ; (5, -1, 5, 5) ; and (-1, 5, 5, 5). There are other neat methods of solution.

(Correct solutions from G. Lewis (Sydney Boys' H. S.); J. Mock (S. B. H. S.); B. Temple (Albury H. S.); T. Weir(Canberra

Grammar); and partly correct solutions from D. Beaumont (Griffith H. S.); D. Hermann (Oak Flats H. S.); J. Milston (Syd. Grammar); and J. Walsh (De La Salle College, Ashfield).)

O16. Show that every "map" obtained by drawing a number of complete circles which overlap in any fashion may be fully coloured with two colours in such a way that regions with a positive length of boundary in common have different colours.

Answer. Colour the map according to the following rule;- If a region lies inside an even number of circles (or 0 circles) use colour A; if it lies inside an odd number of circles use colour B. Every region is certainly coloured with one of the colours. Adjacent regions are separated by a circular arc, one region lying inside the circle, and one outside it. Hence the numbers of circles in which the adjacent regions lie differ by one, and they will be given different colours by the rule.

(The problem can also be tackled by mathematical induction).

(Partly correct solutions received from G. Lewis and B. Temple)

O17. P is a simple polyhedron with more than eight faces. Each face has n edges, and r edges meet at each vertex. Show that P has exactly 30 edges.

ANSWER. Since P is a simple polyhedron, $F + V - E = 2$ (Euler's Formula; F is the number of faces, V the number of vertices, and E the number of edges).

Now , counting the edges face by face, or vertex by vertex we obtain $Fn = 2E$ and $Vr = 2E$. Substituting for F and V in Euler's Formula gives $2E/n + 2E/r - E = 2$. Hence $1/n + 1/r = 1/2 + 1/E$ (1)

It is obvious that n and r are integers not less than three. If they both exceed three $1/n + 1/r \leq 1/4 + 1/4 = \frac{1}{2} < 1/2 + 1/E$, contradicting (1). Hence either $n = 3$, or $r = 3$.

Case 1 $n = 3$.

(1) gives $1/r = 1/6 + 1/E$. There are three possible values of r, viz $r = 3$, $r = 4$, or $r = 5$. These yield polyhedra with $(n, r, F, V, E) = (3, 3, 4, 4, 6)$; $(3, 4, 8, 6, 12)$ or $(3, 5, 20, 12, 30)$ respectively. Only the last of these hss more than 8 faces, and it has 30 edges.

Case 2 $r = 3$.

Identical working (with n and r interchanged and F and V interchanged shows that polyhedra with (n, r, F, V, E) equal to $(4, 3, 6, 8, 12)$; or $(5, 3, 12, 20, 30)$ satisfy (1), and only the second of these has more than eight faces. It has 30 edges.

(Correct solutions from G. Lewis (S. B. H. S.) and B. Temple (Albury H. S.).)

O18. $P(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ is a polynomial with integral coefficients $a_1, a_2, \ldots\ldots a_n$. For some integer b, $P(b)$ and $P(b + 1)$ are both odd numbers. Prove that the equation $P(x) = 0$ has no integral solutions.

Answer. Suppose on the contrary that for an integer k, $P(k) = 0$. Then by the factor theorem

$P(x) = (x - k)F(x)$, where $F(x)$ is a polynomial of degree $n - 1$ with integral coefficients. Then $P(b) = (b - k)F(b)$ and $P(b + 1) = (b + 1 - k)F(b + 1)$, and one of the consecutive integers $b - k$ and $b + 1 - k$ is an even number. It follows that either $P(b)$ or $P(b + 1)$ is even, contradicting the data.

(Correct solutions from T. Weir (Canberra Grammar), and G. Lewis (S. B. H. S.).)

O19. In chess, is it possible for the knight to start at the lower right hand corner of the board and finish at the upper left hand corner, lighting exactly once on every square on the board?

Answer. No. Suppose the lower R. H. corner is a white square. If such a knight's tour were possible, the knight would make 63 moves before reaching the L. H. top corner, another white square. But the colour of the square occupied by the knight changes with each move. After an odd number of moves it must reach a black square.

(Correct solutions from J. Walsh (De La Salle Coll., Ashfield), W. J. Miller (Newington College), J. Milston (Syd. Grammar), J. Mock (S. B. H. S.) G. Lewis (S. B. H. S.) and B. Temple (Albury H. S.).)

O20. Four brothers A, B, C, and D, sold a herd of cattle, each beast fetching as many pounds as there were animals in the herd. They shared the money in the following way. First A took £10, then B took £10, then C, then D, then A again, and so on. Eventually, it being D's turn, there was less than £10 remaining. However, A, B, and C each returned a sum of money to D so that the division was perfectly fair. How much did each return?

Answer. Let the number of animals in the herd be $20k \pm x$, where k and x are integers and $0 \leqslant x < 10$ (so that $x^2 < 100$). The sum of money obtained by the sale of the herd is £$(20k \pm x)^2$, i.e. £$(400k^2 \pm 40kx + x^2)$. Since $(400k^2 \pm 40kx)$ is divisible by 40, a stage will be reached in the sharing out when £x^2 remains, it being A's turn to take £10. It is clear from the data that x^2 lies between 30 and 40, or else between 70 and 80. The only possibility is $x^2 = 36$. Only £6 remains when it is D's final turn, The others must return £1 each to D, reducing their last share from £10 to £9, and increasing D's from £6 to £9.

(Correct solutions from B. Temple (Albury H. S.); J. Milston (Syd. Grammar); G. Lewis (S. B. H. S.). Partly correct solutions from D. Hermann, J. Mock, T. D. Potts, and W. J. Miller).

ASSORTED APOLOGIES

We regret that the names of the following students who submitted correct answers to problems O5 and O10 were accidentally omitted from the last issue.

O5. Correct solutions from J. Lloyd, T. Ryall, and G. Owerchuk.

O10. Correct solutions from K. Schofield, G. N. Epstein, J. Walsh, G. McDonnell, F. Hutchinson, G. Owerchuk, and S. Polhill.

Prepared. Several people (including one student-B. Sterret) have pointed out, and no doubt several others noticed, that the solution we published to the puzzle at the foot of p. 12 of the last issue didn't work. Indeed it is easy to calculate that the planks had to be at least 9·43 ft long to span the moat when arranged as in our diagram. If that boy scout was too puny to jump 10 ft he just had to be prepared to swim.

OWERCHUK'S PROBLEM

As no one has yet made any contribution towards solving this problem (Parabola, Vol. 1, No 2, p. 10), we will defer further comment until a later issue of Parabola.

PUZZLE. Replace the letters by digits so that the two subtractions are simultaneously correct.

```
  N I N E       N I N E
-   O N E     -   T E N
  -------       -------
    A L L         T W O
```

(Answer, p. 20)

ANSWERS.

CUBES. (p. 2) No two small cubes are identical.

HORSE SENSE. Put one horse in each of four stalls, and sixteen horses in the fifth stall.

No objections, please. Sixteen is a very odd number of horses to put in one stall.

RISE OR FALL. (p. 11) The level of water falls.

SIMPLE ARITHMETIC. (p. 12)

```
            8 0 8 0 9
124 ) 1 0 0 2 0 3 1 6
        9 9 2
          1 0 0 3
            9 9 2
              1 1 1 6
              1 1 1 6
```

PUZZLE. (p. 19)

L = 0;	A = 5;	1319	1319
N = 1;	T = 6;	819	691
W = 2;	O = 8;	500	628
I = 3;	E = 9.		
