UNSOLVED PROBLEMS

In the October 1964 issue of Parabola, the article on the Four Colour Problem called your attention to the existence of numerous unsolved mathematical problems which can be stated in quite simple non-technical terms. From time to time we wish to write about such "elementary" unsolved problems.

In the same issue, one of our readers, G. Owerchuk asked what is the largest number n_r of points which may be completely connected with coloured line segments using r different colours, in such a way that no one colour triangle results.

For the special case $r = 2$ the problem was set in a previous issue of Parabola (Vol.1, No. 1 Problem 0_7) and readers were asked to prove $n_2 = 5$. Apart from a few small values of r ($n_1 = 2$, $n_2 = 5$, $n_3 = 16$) the general solution of the problem is unknown. It can be shown by the same method as used in the solution of 0_7 that

$$(1) \qquad n_r \leq r\, n_{r-1} + 1$$

For let c_1, c_2, ..., c_r be the r given colours, p_0 one of the given points, and s_1 the set of points connected to p_0 by a line segment of colour c_1. No two points $p_1 p_2$ of the set s_1 can be connected by colour c_1, since otherwise $p_0 p_1 p_2$ would be a one colour triangle. Hence the segments connecting points of s_1 can have only $r - 1$ colours, namely c_2, c_3, ..., c_r. Since they are not supposed to form a triangle of the same colour, s_1 contains at most n_{r-1} points. The same is true of s_2, the set of points connected to p_0 with a segment of colour c_2, etc. Thus the sets s_1, s_2, ...s_r contain altogether at most $r\, n_{r-1}$ points. Since these points together with p_0 exhaust all points of the diagram, we obtain the required inequality.

From the inequality it follows by mathematical induction that

$$(2) \qquad n_r \leq r!\ \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{r!}\right)$$

In fact, for $r = 1$, the inequality becomes $n_1 \leq 1!(1 + \frac{1}{1!}) = 2$, which is true since obviously $n_1 = 2$. Now let $r > 1$ and suppose that we have already proved that

$$(3) \quad n_{r-1} \leqslant (r-1)!(1 + \frac{1}{1!} + \dots + \frac{1}{(r-1)!})$$

Then the inequalities (1) and (3) give

$$\begin{aligned} n_r &\leqslant r\, n_{r-1} + 1 \\ &\leqslant r(r-1)!(1 + \frac{1}{1!} + \dots + \frac{1}{(r-1)!}) + 1 \\ &= r!(1 + \frac{1}{1!} + \dots + \frac{1}{(r-1)!} + \frac{1}{r!}), \end{aligned}$$

which proves (2).

For $r = 2$, we have $n_2 \leqslant 2!(1 + \frac{1}{1!} + \frac{1}{2!}) = 5$ and for $r = 3$

$$n_3 \leqslant 3!(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}) = 16.$$

In both cases there is equality, $n_2 = 5$, $n_3 = 16$. To verify these values it is sufficient to produce a configuration of the required kind with 5 points and 2 colours, or 16 points and 3 colours. An example with 5 points and 2 colours was reproduced in Parabola Vol. 1 No. 2. The following is a configuration with 16 points and 3 colours, due to Mr. Cox.

We designate the 16 points by

$$\begin{matrix} p_0 & q_1 & q_2 & q_3 & q_4 & q_5 \\ & r_1 & r_2 & r_3 & r_4 & r_5 \\ & s_1 & s_2 & s_3 & s_4 & s_5 \end{matrix}$$

The following segments are given the colour c_1,

$$
\begin{array}{lllll}
p_0\ q_1, & p_0\ q_2, & p_0\ q_3, & p_0\ q_4, & p_0\ q_5, \\
s_1\ s_2, & s_2\ s_3, & s_3\ s_4, & s_4\ s_5, & s_5\ s_1, \\
r_1\ r_3, & r_3\ r_5, & r_5\ r_2, & r_2\ r_4, & r_4\ r_1, \\
r_1\ s_1, & r_2\ s_2, & r_3\ s_3, & r_4\ s_4, & r_5\ s_5, \\
q_1\ r_3, & q_2\ r_4, & q_3\ r_5, & q_4\ r_1, & q_5\ r_2, \\
q_1\ r_4, & q_2\ r_5, & q_3\ r_1, & q_4\ r_2, & q_5\ r_3, \\
s_1\ q_4, & s_2\ q_5, & s_3\ q_1, & s_4\ q_2, & s_5\ q_3, \\
s_1\ q_5, & s_2\ q_1, & s_3\ q_2, & s_4\ q_3, & s_5\ q_4.
\end{array}
$$

Those given the colour c_2 are obtained by replacing q_i by r_i, r_i by s_i and s_i by q_i in each entry of the previous scheme, and those given the colour c_3 are obtained by a similar cyclic replacement of q_i by s_i, r_i by q_i, s_i by r_i.

Because of the perfectly symmetrical and cyclic nature of the construction it is quite sufficient to verify that no c_1 triangle exists in which one of the segments is $p_0\ q_1$, or $s_1\ s_2$, or $r_1\ r_3$, or $r_1\ s_1$, or $q_1\ r_3$, or $q_1\ r_4$, or $s_1\ q_4$, or $s_1\ q_5$. This can be done quite easily by inspection of the diagram.

For $r > 3$ nothing definite is known about the problem. It may be conjectured that for every r

$$n_r = r!(1 + \frac{1}{1!} + \ldots + \frac{1}{r!})$$

that is, there exists a configuration with that many points when r colours are used. For $r = 4$, there is a known configuration with 41 points (published in 1955 by Greenwood and Gleason in the Canadian Journal of Mathematics), but whether this can be bettered to the theoretically best possible value, 65, is not known.

In the next issue we shall deal with another famous unsolved problem on configurations.

Professor G. Szekeres.

ANSWERS TO THE PUZZLES

ON SAFARI (p.7)

One solution is :- 1A:CC; 1B:C; 2A:CC; 2B:C; 3A:MM; 3B:MC; 4A:MM; 4B:C; 5A:CC; 5B:C; 6A:CC.

To decipher this solution read, for example, 3B:MC as "On the third trip back the canoe carries one missionary and one cannibal", and 4A:MM as "On the fourth trip across the canoe carries two missionaries". The first two and the last two trips may be varied in an obvious way.

DOMINOES (p.16)

No. Since the two squares to be left uncovered are both of the one colour, we are required to cover 30 squares of that colour and 32 squares of the other colour, which is obviously impossible since each domino covers one white square and one black one.

NOT SO OBVIOUS (p.32)

$$\begin{array}{r} 29786 \\ +\ \ 850 \\ +\ \ 850 \\ \hline 31486 \end{array}$$