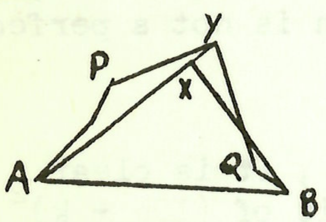
ANSWERS TO THE JUNIOR COMPETITION QUESTIONS.

Q.1. X is a point inside a polygon and AB is one of the sides of the polygon. Show that the perimeter of triangle ABX is shorter than the perimeter of the polygon.
[You may assume that the shortest distance between two points is a straight line segment].

ANSWER:



Let AX produced cut the perimeter of the polygon at Y .

Then AX + XY < length APY .

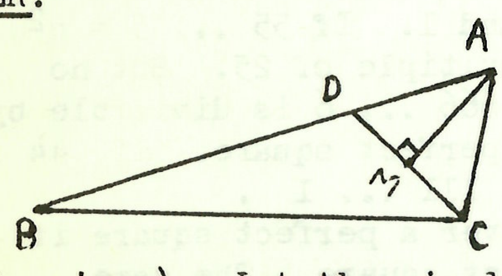
and BX < XY + length YQB .

Adding AX + BX + XY < length APYQB + XY .

If we subtract the length XY from both sides of this inequality and add the length AB , we obtain the required result.

Q.2. Show how to construct a triangle given the angle A , the length of the opposite side, and the difference between the lengths of the other sides.

ANSWER:



Construct the accompanying diagram in the following order. Construct line segment BD whose length is the given difference of the lengths of the sides meeting at A . Construct the line DC making $BDC = 90^\circ + \frac{A}{2}$ (A is given). Let the circle whose centre is B and whose radius is the given length of the side opposite A intersect the line DC at C . Construct the perpendicular bisector of DC and produce BD to intersect this at A . Join AC .

It is readily proved that the ΔABC has the specified properties. Thus (i) the side BC opposite A has been constructed equal to the given length. (ii) Since $\Delta AMD \equiv \Delta AMC$ (2 sides and included angle theorem). It follows that AD = AC and hence that AB - AC = BD; and this has been constructed equal to the required difference of lengths of AB and AC . (iii) Finally $\angle CAM = \angle DAM = \angle BDM - \angle DMA$ (ext angle = sum of interior opposites) $= 90^\circ + \frac{A}{2} - 90^\circ = \frac{A}{2}$.

(Answer - Q.2. Cont'd)

Hence: $\angle BAC = \angle CAM + \angle DAM = \frac{\angle A}{2} + \frac{\angle A}{2} = \angle A$, (the given angle).

Q.3 (i) Prove that a whole number which is a perfect square cannot end with the digit 7.

(ii) A uni-digit number is one which consists only of one digit repeated a number of times. Examples are 888 or 33 or 555, 555. Show that a uni-digit number greater than ten is not a perfect square.

ANSWER:

(i) Since $(10k + k)^2 = 10(10h^2 + 2hk) + k^2$, it is clear that the last digit of k^2 , is also the last digit of $(10h + k)^2$. Since none of the squares

$$0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 \text{ or } 9^2$$

end in a 7, neither can the square of any other integer.

(ii) It is clear from (i) that we can immediately dismiss the possibility of 77 ... 7 being a perfect square, and the same argument applies for the digits 2, 3, and 8. Since 0 is trivial, we need consider only the digits 5,6,4,9 and 1. If $55 \ldots 5 = q^2$ then q is a multiple of 5, and q^2 a multiple of 25. But no multiple of 25 ends in ... 55. Similarly 666 ... 6 is divisible by 2 but not by 4, and therefore cannot be a perfect square. If $44 \ldots 4 = q^2$, q is even, $q = 2Q$ and $Q^2 = 111 \ldots 1$. Hence, if we can show that 1 1 ... 1 is never a perfect square it will follow that 44 ... 4 is not a perfect square. The same argument will apply to $999 \ldots 9 = 3^2 (111 \ldots 1)$.

Thus we are finished if we show that 111 ... 1 is not a perfect square. Suppose $q^2 = 111 \ldots 1$, then q is odd, $q = 2n + 1$, and $q^2 = 4(n^2 + n) + 1$. i.e q^2 leaves a remainder of 1 when divided by 4. But $111 \ldots 11 = 111 \ldots 100 + 8 + 3$ which obviously leaves a remainder of 3 on division by 4 , and therefore cannot be the square of an odd number.

This completes the proof.

Q.4. a,b and c are three numbers such that $a > b > c$.

(i) Prove that -

$$\frac{1}{a - b} + \frac{1}{b - c} + \frac{1}{c - a} > 0$$

Q.4 Cont'd

(ii) Is the inequality -

$$\frac{1}{a-b} + \frac{1}{b-c} + \frac{3}{c-a} > 0$$

also true for these numbers?

ANSWER: Put $a - b = h$, $b - c = k$. Then h and k are positive, and $h + k = a - c$.

(i) $$\frac{1}{a-b} + \frac{1}{b-c} + \frac{1}{c-a} = \frac{1}{h} + \frac{1}{k} - \frac{1}{h+k}$$

$$= \frac{k(h+k)+h(h+k)-hk}{hk(h+k)}$$

$$= \frac{h^2+hk+k^2}{hk(h+k)}$$

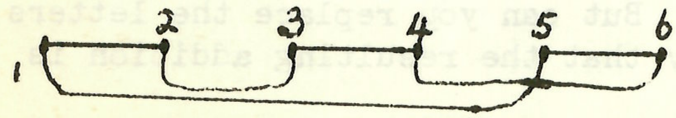
which is obviously positive (every term is positive on numerator and denominator).

(ii) Yes. L.H.S. $$= \frac{k(h+k)+h(h+k)-3hk}{hk(h+k)}$$

$$= \frac{h^2-hk+k^2}{hk(h+k)} = \frac{(h-k)^2+hk}{hk(h+k)}$$

and since $(h-k)^2 \geqslant 0$ the numerator is again positive.
[If the 3 is replaced by 4 the expression vanishes if $h = k$ (ie. if $b = \frac{a+c}{2}$) and is otherwise positive; if the 3 is replaced by any larger number than 4, the expression may be either positive or negative].

Q.5



Six points numbered 1 to 6 are connected by a line (which may cross itself) to form a single circuit; that is starting at 1 and moving along the line we pass through each point once and only once before reaching 1 again. If part of the circuit includes direct connections

Q.5 Cont'd:

between 1 and 2, between 3 and 4 and between 5 and 6, in how many different ways may the circuit be completed? (The diagram indicates one way).

If a single circuit is to be drawn through ten points, to include five direct connections, those between 1 and 2, between 3 and 4, between 5 and 6, between 7 and 8 and between 9 and 10, in how many different ways may this be done?

ANSWER: (i) The point 1 may be joined to any one of 4 points (3,4,5 or 6). However this is done, there are two points to which 2 may be connected, and then only one way of completing the circuit. Hence there are 4 x 2 different ways of obtaining a circuit.

(ii) The point 1 may be connected to anyone of 8 points. Suppose it is joined to point x which is directly connected to point $y(= x \pm 1)$. Then y may be further connected to any one of 6 points, the end points of the 3 segments which have not yet been put into the circuit. The other end of whichever segment is chosen may then be joined to any one of the 4 remaining points, and then there remain 2 ways of completing the circuit by introducing the one remaining segment. There are altogether $8 \times 6 \times 4 \times 2$ different ways of making a circuit (i.e. 384 ways).

NOT SO OBVIOUS.

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      F O R T Y
  +       T E N
  +       T E N
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      S I X T Y
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This looks obvious. But can you replace the letters by digits in such a way that the resulting addition is still correct?

(Answer on p 13)