Parabola Volume 2, Issue 2 (1965)

SOLUTIONS OF PROBLEMS IN PARABOLA, VOL. 2, NO. 1.

J21. (i) In a box there is a number of balls of c different colours. What is the smallest number of balls for which we can say that, however the colours are distributed, there is at least one set of s balls of the same colour? Justify your statement.

(ii) Further, if $n > 0$, what is the smallest number of balls for which we can say that there are at least n sets, each set containing s balls of the one colour? (In forming these sets we do not require that sets be of different colours; e.g. from 2s balls coloured red we could form 2 sets).

Answer. (i) The required number is $c(s-1) + 1$. If we draw out only $c(s-1)$ balls it is possible that we might have exactly $(s-1)$ balls of each of the c colours. In this case, however, drawing out one more ball, whatever its colour, will raise the number of balls of that colour to s .

(ii) The required number is $c(s-1) + 1 + (n-1)s$. Since this number is not less than $c(s-1) + 1$ (n is a positive integer) we can certainly select 1 set of s balls of the same colour by (1). After this has been done $(n-1)$ times, exactly $c(s-1) + 1$ balls remain and, again by (1), an nth set of s balls of the same colour can be selected. If, however, any smaller number of balls is withdrawn, even only one fewer, $c(s-1) + (n-1)s$, we cannot be sure of having n sets of s balls of the one colour. For example, we may have withdrawn $(n s - 1)$ red balls and $(s-1)$ balls of each of the remaining $(c-1)$ colours. (Note that $(s-1)(c-1) + (n s - 1) = c(s-1) + (n-1)s$). After selecting s red balls $(n-1)$ times, we are left with $(s-1)$ balls of each of the c colours, and no nth set of s balls can be selected.

B. Ratcliffe, Marist Bros, Eastwood, stated correct results.

J22. (i) There are seven points on a straight line in the order A, B, C, D, E, F, G and P is a variable point on the line. For what position of P is the sum of the positive lengths

$$PA + PB + PC + PD + PE + PF + PG$$

least? Justify your statement.

(ii) Answer a similar question for six given points U, V, W, X, Y, Z on a line, and a variable point Q.

Answer. (i) The sum of the lengths is least when P coincides with D. Indeed for any position of P

$PA + PG \geqslant AG = DA + DG$. (the equal sign holds if P is in the interval AG; otherwise the $>$ sign holds)

$PB + PF \geqslant BF = DB + DF$.

$PC + PE \geqslant CE = DC + DE$.

$PD \geqslant O = DD$.

Adding. $PA + PB + PC + PD + PE + PF + PG \geqslant AG + BF + CE = DA + DB + DC + DD + DE + DF + DG$.

(ii) Exactly similar reasoning shows that provided Q lies anywhere on the interval WX, the value of $QU + QV + QW + QX + QY + QZ = UZ + VY + WX$. If it lies outside this interval the L. H. S is greater.

Correct solution from B. Gaynor, Marist Bros. Eastwood.

J23. (a) Prove that if n is a positive integer greater than 2, the numbers $2^n + 1$ and $2^n - 1$ are not both prime.

(b) Prove that of the three positive integers a, $8a - 1$ and $8a + 1$, there is at least one which is not prime.

Answer. Of any 3 consecutive integers one is divisible by 3.

Answer (cont.) (a) Take the 3 consecutive integers $2^n - 1$, 2^n, $2^n + 1$. Since, $n > 2$, the smallest is > 3. Since 2^n is not divisible by 3, one of the others is, and this one therefore cannot be prime.

(b) Take the 3 consecutive numbers $8a - 1$, $8a$, and $8a + 1$. If 3 does not divide a, it does not divide 8a, and therefore either $8a - 1$, or $8a + 1$ is a multiple of 3 and not prime.

If a is a multiple of 3, then it is not prime unless it is equal to 3 in which special case $8a + 1$ (25) is composite.

Correct solutions from D. O'Connor, St. Mary's College, Grafton. D. W. Hales, Canterbury Boys' High School.

J24. All the odd integers, beginning with 1, are written successively (that is, 1 3 5 7 9 11 13 15 17 19 21...). Which digit occupies the 100,000th position?

Answer. The first 5 positions are filled by the 5 odd 1 digit numbers. The next 90 positions are filled by the 45 odd 2 digit numbers. The next 1350 positions are filled by the 450 odd 3 digit numbers. The next 18,000 positions are filled by the 4,500 odd 4 digit numbers. The remaining 80,555 positions are filled by the 16,111 odd 5 digit numbers from 10,001 to 42,221 inclusive. The last digit of this number, (1) is therefore the digit which occurs in the 100000 th position in the list.

(All who sent solutions misconstrued the question to mean "which is the 100,000th odd integer?"; to which they gave the correct answer, 199,999. Our apologies if the question wasn't as clear as it might have been).

O25. (i) If S_r denotes the sum of the rth powers of the numbers 1, 2, 3, . . . , 9, show by considering the sign of

$$(1 + x)^2 + (2 + 4x)^2 + (3 + 9x)^2 + \ldots + (9 + 81x)^2$$

that $S_3^2 < S_2 S_4$.

Give an outline proof also of the following:-

(ii) $S_2^2 < 9S_4$;

(iii) $S_{m+n}^2 < S_{2m} S_{2n}$, if $n \neq m$.

(iv) State and justify - in a few lines of discussion - more general results applicable to sets of numbers other than 1, 2, . . . , 9.

Answer. (i) $$\sum_{k=1}^{9} (k + k^2x)^2 = (1 + x)^2 + (2 + 4x)^2 + \ldots + (9 + 81x)^2$$

$$= (1^2 + 2^2 + \ldots 9^2) + 2(1.1 + 2.4 + \ldots + 9.81) x + 1 (1^2 + 4^2 \ldots + 81^2) x^2.$$

$$= S_2 + 2 S_3 x + S_4 x^2.$$

This quadratic expression, being the sum of squares, is always positive (its value is never zero, since x cannot be chosen to make all terms vanish simultaneously). Hence the discriminant of the quadratic is negative i. e. $4S_3^2 - 4S_2 S_4 < 0$, or $S_3^2 < S_2 S_4$.

(ii) Put m = 0, n = 2 in the result (iii).

(iii) $$\sum_{k=1}^{9} (k^m + k^n x)^2 = S_{2m} + 2 S_{m+n} x + S_{2n} x^2$$

The discriminant of this quadratic expression, $4S_{m+n}^2 - 4S_{2m} S_{2n}$, is negative since the quadratic expression is always positive.

Answer (cont.) (iv) Let $a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots b_N$ be any two sets of N real numbers.

If $\frac{a^r_1}{b^s_1}, \frac{a^r_2}{b^s_2}, \ldots, \frac{a^r_N}{b^s_N}$, are not all equal, x cannot be chosen so that $(a^r_1 + b^s_1 x), (a^r_2 + b^s_2 x), \ldots, (a^r_N + b^s_N x)$ all vanish together.

Hence $\sum_{k=1}^{N} (a^r_k + b^s_k x)^2 = \sum_{k=1}^{N} a^{2r}_k + 2\left(\sum_{k=1}^{N} a^r_k b^s_k\right) x + \left(\sum b^{2s}_k\right) x^2$ is positive definite; and therefore has a negative discriminant

i.e. $\left(\sum_{k=1}^{N} a^r_k b^s_k\right)^2 < \left(\sum_{k=1}^{N} a^{2r}_k\right) \left(\sum_{k=1}^{N} b^{2s}_k\right)$.

Correct solutions from F. Hutchinson (Marist Bros. Eastwood, G. Lewis, S. B. H. S

O26. We say that a plane is transformed by a similitude, of centre O (on the plane) and ratio k, if each point P on the plane is transformed into a point P' on the line OP, such that

OP' = k. OP.

If k is negative, the senses of the segments OP and OP' are opposite.

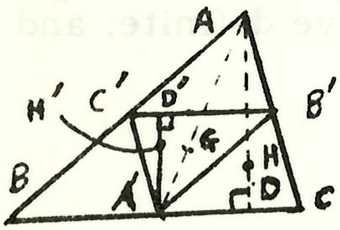
(i) ABC is a triangle whose centroid is G. Show that, under a similitude of centre G and ratio $k = -\frac{1}{2}$, the points A, B, C transform

O26. (cont.)

into A', B', C', the mid-points of the sides. Hence show that H, G, and S are collinear, H being the orthocentre and S the circumcentre of the triangle ABC. If, on the line HGS, G is taken as origin and S has co-ordinate - 1, what is the co-ordinate of H?

(ii) If the lines drawn parallel to BC, CA, AB through A, B, C respectively meet in A_1, B_1, C_1, show that H_1, the orthocentre of the triangle $A_1 B_1 C_1$, lies on HGS. What is its co-ordinate in the above system?

Answer. (i) Note that similitudes transform any straight line into a parallel straight line and hence preserve the angle between two lines.



Since the centroid G, is a point of trisection of the medians, i. e. if M is the mid point of BC, GM = - $(\frac{1}{2})$ GA, it is clear that M coincides with A'. Thus A', B', C' are the mid points of the sides ABC

Since angles are preserved by a similitude, the altitude AD of ABC transforms into an altitude A' D' of A' B' C', and hence H the orthocentre of ABC transforms into the intersection of the altitudes (i. e. the orthocentre) H' of A' B' C'. i. e. H' G H is a straight line and GH' = - $\frac{1}{2}$. GH.

But since B' C' $\parallel$ BC, A' D' is not only an altitude of A' B' C', but also the perpendicular bisector of BC. Thus S, the circumcentre of ABC, coincides with H'.

Since GH = - 2 GH' = - 2 GS, if GS = - 1 then GH = + 2, and this is the co-ordinate of H on the line HGS.

(ii) It is easy to see that G is the centroid of $A_1B_1C_1$.

Answer (cont.) Hence, by the reasoning in (i), the given similitude transforms the orthocentre H_1 of $A_1B_1C_1$ into the orthocentre H of ABC i.e. H_1 lies on the line HG and $GH = -\frac{1}{2} GH_1$; whence $GH_1 = -2x\ GH = -2x + 2) = -4$.

Correct solution from G. Lewis (S. B. H. S.)

O27. Consider all points in the co-ordinate plane XOY whose co-ordinates are both positive integers. The sum x + y of the co-ordinates of such a point is called the index of the point. I write down the point (1,1) of index 2; then the points (1,2) (2,1) of index 3 in ascending order of their first co-ordinates; then the points (1,3), (2,2), (3,1) of index 4 in ascending order of their first co-ordinates; and so on. The result is:- (1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (1,4), (2,3), (3,2), (4,1), (1,5),... .

If (x, y) is the nth point written down, find a formula for n in terms of x and y.

Answer. The points of index k are (1,k-1), (2,k-2), ...(k-1,1); (k-1) points in all. The number of points with index $\leq$ k is therefore $1 + 2 + 3 + \ldots + (k-1) = \frac{1}{2} k (k-1)$. Since (x, y) has index x + y, the number of points with a smaller index is obtained by putting k = x + y - 1 in this formula. All of these points precede (x, y). Also (x, y) is the xth point of index x + y. Hence (x, y) occurs in the nth position where $n = \frac{1}{2}(x + y - 1)\cdot(x + y - 2) + x$.

Correct solutions from J. Kalman, (St. Ignatius, Riverview), J. Mock (S. B. H. S.) P. Underwood, Manly B. H. S.) W. J. Müller, Newington College, P. Leonard, Eastwood, (Marist Bros.) G. Matheson, Drummoyne, D. Hermann, Oak Flats H. S., G. McDonnell, (St. Pius X, Chatswood), D. Bradfield S. G. E. G. S. (Partly Right), G. Lewis (Inverse problem given n find (x_1y).

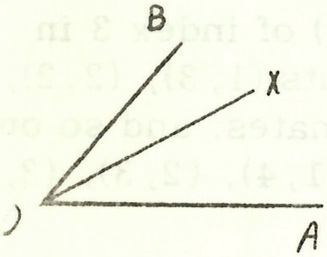
Q28. (a) $\angle$ AOB is an acute angle and OX is a variable ray dividing it into two smaller angles. Show that

$$\sin \angle AOX + \sin \angle BOX$$

is greatest when OX bisects $\angle$ AOB.

(b) A polygon with n sides is inscribed in a circle. Show that the area of the polygon is greatest if it is regular.

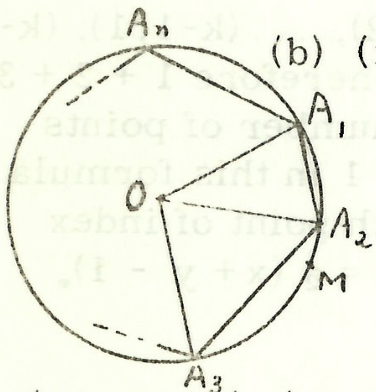
Answer. (a) $\sin \angle AOX + \sin \angle BOX$



$$= 2 \sin \tfrac{1}{2} (\angle AOX + \angle BOX) \cos \tfrac{1}{2} (\angle AOX - \angle BOX)$$

$$= 2. \sin \tfrac{1}{2} \angle AOB. \cos \tfrac{1}{2} (\angle AOX - \angle BOX)$$

Only the last factor is variable and its maximum value of 1 occurs when $\angle AOX = \angle BOX$. The proof applies equally well if $90^\circ \leqslant \angle AOB \leqslant 360^\circ$.



(b) (i) We show that, given any n-sided polygon inscribed ir a circle, we can construct another with larger area unless the first is regular. In fact, if $A_1 A_2 \ldots A_n$ is not regular, the angles $\angle A_1 0 A_2$, $\angle A_2 0 A_3$, . . . , $\angle A_n 0 A_1$ are not all equal. We can find an adjacent pair of these angles which are unequal. We may sup $\angle A_1 0 A_2 \neq \angle A_2 0 A_3$. Then area $\Delta A_1 0 A_2$ + area $\Delta A_2 0 A_3 = \frac{1}{2} r^2$. $(\sin \angle A_1 0 A_2 + \sin \angle A_2 0 A_3)$ and by (a) this is less than $\frac{1}{2} r^2 (\sin \angle A_1 0 M + \sin \angle M_2 0 A_3)$ where M is the mid point of the arc $A_1 A_3$. The polygon $A_1 M A_2 \ldots A_n$ has a larger area than $A_1 A_2 A_3 \ldots A_n$. This completes the proof.

Answer (cont.) There is the same "well-concealed logical error" in this "proof", as that in Steiner's proof of the Isoperimetric theorem, discussed by Professor Szekeres in Parabola, Vol. 1 No. 1. viz, we have proved only that if there is an inscribed n-polygon of maximum area, it must be the regular polygon. To complete the proof we should have to demonstrate that the underlined statement does hold. Instead of attempting to do this, we give the following alternative solution which avoids the difficulty.

(ii) We generalise part (a) by proving the

Lemma:- If $x_1 + x_2 + \ldots + x_n = \alpha$ $(0 < \alpha \leq 2\pi,\ 0 < x_i)$ then

$\sin x_1 + \sin x_2 + \ldots + \sin x_n$ is maximum when $x_1 = x_2 = \ldots = x_n = \frac{\alpha}{n}$.

Proof. We show this by induction on n. Assume it is true for $n = k - 1$. After choosing x_1 in any fashion, $\sin x_2 + \ldots + \sin x_k$ is maximum when $x_2 = x_3 = \ldots = x_n = \frac{\alpha - x_1}{k - 1}$, by the induction hypothesis. Hence $\max(\sin x_1 + \sin x_2 + \ldots \sin x_k) = \max\left(\sin x_1 + (k-1)\sin\left(\frac{\alpha - x_1}{k-1}\right)\right)$

We can find the maximum value of this function by using calculus.

Set $y = \sin x_1 + (k-1)\sin\frac{\alpha - x}{k-1}$

then $\frac{dy}{dx_1} = \cos x_1 - \cos\frac{\alpha - x_1}{k-1}$

This vanishes when $x_1 = \frac{\alpha - x_1}{k-1}$, $x_1 = \frac{\alpha}{k}$ and for this value of x_1

$\frac{d^2y}{dx_1^2} = -2\sin\frac{\alpha}{k} < 0$, so that the stationary value is a maximum.

Hence the truth of the Lemma for $n = k$ follows from its truth for $n = k - 1$, and since in (a) we have proved it when $x = 2$, the Lemma must be true when $n = 3, 4$, in fact, any positive integer.

Answer (cont.) Since the area of an inscribed n-polygon whose sides subtend angles $x_1, x_2, \ldots x_n$ at the centre (so that $x_1 + x_2 + \ldots x_n = 2\pi$) is equal to $\frac{1}{2} r^2 (\sin x_1 + \sin x_2 + \ldots + \sin x_n)$, it immediately follows that the n-polygon of maximum area is that when $x_1 = x_2 = \ldots = x_n = \frac{2\pi}{n}$, i. e. the regular polygon.

Correct solutions from P. A. Collins, Marist Bros. Eastwood, W. J. Müller, Newington College, D. Bradfield, S. C E G. S., F. Hutchinson, Marist Bros. Eastwood, G. Lewis, S. B. H. S. P. Underwood (Manly B H. S.) J. Mock, S. B. H. S.

O29. Two people, A and B play a game in which they alternately remove at least one and not more than three matches from a heap. A makes the first move. The player who is forced to take the last match loses. If there are initially 60 matches in the heap, which player should win? What is his correct strategy?

Answer. A will win if he employs the following strategy. On his first move he takes 3 matches, leaving $57 = 4 \times 14 + 1$. If then B takes x matches, A replies by taking $4 - x$ matches, leaving $4 \times 13 + 1$ matches. A continu es to use this method of play and, after his 15th turn there remains only 1 match which B is forced to take.

Correct solutions from J. Kalmar Riverview, P. A. Collins Marist Bros. Eastwood, J. Mock S. B. H. S., G. Thomas Marist Bros. Eastwood, W. J. Müller Newington, D. Bradfield S. C. E G. S., B. Goodman Marist Bros. Eastwood, C. Binnie Marist Bros. Eastwood, G. Matheson Drummoyne, G. Lewis S. B. H. S., D. Hermann Oak Flats H. S., G. McDonnell, St. Pius X Chatswood, P. Underwood Manly B. H. S., F. Hutchinson Marist Bros. Eastwood, (Partly Right, B. Fraser, Toronto H. S. and P. Sky Knox Grammar School.)

030. (P) In a certain garden there are a number of beds each planted with flowers.

(Q) If x and y are two different varieties of plants growing, there is exactly one bed containing both.

(R) If any bed is considered there is one and only one other bed in which none of the same varieties of plants are growing.

Show that (i) every bed has at least 2 varieties of plants.

(ii) there are at least 4 varieties of plants in the garden.

(iii) there are at least 6 flower beds;

(iv) no bed has more than 2 varieties of plants.

Answer. (i) Suppose a bed A contains only 1 variety of plants, x. Then there is exactly one bed, B, in which x does not grow (by R). Let B contain variety y (-it must have at least one variety by P). Then there is a bed C containing both x and y (by Q). Since B is the only bed not containing x, and it contains y, there is no bed containing none of the varieties in C , and this contradicts R .

(ii) Any bed A contains at least 2 varieties, x and y, say, by (i). There exists (by R) another bed B containing none of these varieties, but at least 2 others (w and z say) by (i). Hence there are at least 4 varieties.

(iii) In addition, to the beds A and B in (ii) there must exist a bed containing w and x, one containing w and y one containing z and x, and one containing z and y, (Q). Also, no two of these beds can coincide (by Q). Hence there are at least 6 beds.

(iv) First note that given any bed A, there is a unique bed B having none of the same varieties (by R); and therefore that any other bed C has at least one variety which occurs in A , and not more than one (by Q). Similarly C has exactly one variety which occurs in B.

Answer (cont.) If A contains n varieties ($x_1, x_2, \ldots x_n$) and B contains m varieties ($y, \ldots y_m$) then for each pair ($x_h y_k$) there is one bed (by Q), and for any bed C there is a corresponding pair (by the preceding paragraph). Since there are n m such pairs the total number of beds is (n m + 2). Further each variety x_i occurs in exactly m beds in addition to A; (m + 1) beds in all.

We show that B and C have the same number of varieties. Let the bed D be the unique bed containing none of the varieties in C. There is a variety (x_1 say) which occurs in both A and D; x_1 occurs in (m + 1) beds, as we have seen. But, if C contains c varieties, the same argument shows that x_1 occurs in (c + 1) beds. Hence c = m.

Since it can be similarly proved that C and A contain the same number of varieties, we conclude that all beds contain the same number of varieties.

Let every bed contain n varieties. Our previous argument shows that the total number of beds is $(n^2 + 2)$, and, since each variety occurs in (n + 1) beds, the total number of different varieties is

$$\frac{(n^2 + 2)\,n.}{(n + 1)}$$

Since n and n + 1 have no common factor, this expression is an integer only if (n + 1) divides $(n^2 + 2)$ exactly, $(n^2 + 2) = k\,(n + 1)$ say. Then $3 = (n^2 + 2) - (n^2 - 1)$ $= (n + 1) \cdot (k - n + 1)$.

Hence (n + 1) divides 3 exactly. But $n + 1 \geqslant 3$ by (i).

$\therefore$ n + 1 = 3, and n = 2.

(Partly right) P. A. Collins, Marist Bros. Eastwood, J. Mock, S. B. H. S., W. J. Müller, Newington College, D. Bradfield, S. C. E. G. S., F. Hutchinson, Marist Bros. Eastwood, J. Walsh De LaSAlle College, Ashfield, B. Goodman, Marist Bros. Eastwood, G. Lewis, S. B. H. S. I think no-one submitted a correct solution of part (iv).