DECIMAL EXPANSIONS OF RATIONAL NUMBERS.

This article is concerned with the decimal expansions of numbers of the form $\frac{a}{b}$, where a and b are positive integers with no common factor except 1 , and a is less than b .

For example, $\frac{1}{16} = .0625$, a "terminating decimal" ; $\frac{7}{22} = .31818181 \ldots = .3\overline{18}$, where the bar indicates that the block of digits beneath it is repeated indefinitely. In each of these examples ten is the base or scale of the representation of the numbers, but you are already aware of the fact that other numbers may be used in place of ten.

Thus

$$\frac{3}{5} = \left(.\overline{4125}\right)_7 = \frac{4}{7} + \frac{1}{7^2} + \frac{2}{7^3} + \frac{5}{7^4} + \frac{4}{7^5} + \frac{1}{7^6} + \ldots$$

where the subscript implies that 7 is the base used in the representation of $\frac{3}{5}$ (which incidentally is still called a "decimal expansion" of $\frac{3}{5}$, ignoring the fact that decimal derives from a root meaning ten ; there is no other convenient word).

The calculation of such expansions proceeds exactly as in the scale of ten, except that instead of "bringing down a zero" to the end of the remainder at each stage, one must instead multiply by whatever base is used.

To calculate the decimal expansion of $\frac{5}{13}$ using 3 as the base of the representation one proceeds as follows:-

$$
\begin{array}{lr}
 & .\,101.. \\
13 \;\big) & \overline{5 \times 3 = 15} \\
\times 1 & 13 \\
 & \overline{} \\
 & 2 \times 3 = 6 \\
\times 0 & 0 \\
 & \overline{} \\
 & 6 \times 3 = 18 \\
\times 1 & 13 \\
 & \overline{} \\
 & 5 \\
 & =
\end{array}
$$

$$\therefore \quad \frac{5}{13} = (.\overline{101})_3 \ .$$

Indeed, the above merely sets out in a convenient form the following information:-

$$5 \times 3 = 1 \times 13 + 2 \qquad \therefore \ \frac{5}{13} = \frac{1}{3} + \frac{1}{3}\left(\frac{2}{13}\right),$$

$$2 \times 3 = 0 \times 13 + 6 \qquad \therefore \ \frac{2}{13} = \frac{0}{3} + \frac{1}{3}\cdot\left(\frac{6}{13}\right),$$

$$6 \times 3 = 1 \times 13 + 5 \qquad \therefore \ \frac{6}{13} = \frac{1}{3} + \frac{1}{3}\cdot\left(\frac{5}{13}\right);$$

whence $\frac{5}{13} = \frac{1}{3} + \frac{1}{3}\left(\frac{0}{3} + \frac{1}{3}\cdot\frac{6}{13}\right) = \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^2}\left(\frac{1}{3} + \frac{1}{3}\cdot\frac{5}{13}\right)$

$$= \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{1}{3^3}\left(\frac{5}{13}\right).$$

We now answer some questions concerning such expansions.

Q1. Which fractions $\frac{a}{b}$ have terminating decimals in the scale of 10.? (of S?).

ANSWER: Q1. This occurs if the only prime factors of b are also prime factors of 10 (of S). The converse is also true.

Indeed, if $b = 2^h 5^k$ where, say, $k \geq h$, then

$$\frac{a}{b} = \frac{[2^{k-h} a]}{10^k} .$$

Since the numerator is an integer less than 10^k, it is equal to

$$c_1 10^{k-1} + c_2 10^{k-2} + \dots + c_k ;$$

and then

$$\frac{a}{b} = \frac{c_1}{10} + \frac{c_2}{10^2} + \dots + \frac{c_k}{10^k} = .c_1 c_2 \dots c_k ,$$

a terminating decimal. Conversely if $\frac{a}{b} = (.c_1 c_2 \dots c_k)_{10}$ then $10^k . \frac{a}{b}$ is an integer, whence b is a factor of $10^k = 2^k 5^k$. It follows that b has no prime factors other than 2 and 5 .

Q2. Do all repeating decimals represent rational numbers $\frac{a}{b}$? How can one convert to vulgar fraction form?

ANSWER: We show how to carry out the conversion for $(.3\overline{108})_{10}$

$$\text{Let} \quad x = .3\overline{108}$$

$$10^3 \quad x = 310.8\overline{108}$$

$$\therefore \quad (10^3 - 1) \quad x = 310.5 = \frac{3105}{10^1} \quad \therefore \quad x = \frac{3105}{10^1 . (10^3 - 1)} = \frac{23}{74} .$$

It is clear that the procedure can be carried out for any expansion $(.c_1 c_2 \dots c_k \overline{d_1 d_2 \dots d_n})_S$ and will give

$x = \frac{A}{S^k(S^n-1)}$ for some integer A .

Q3. Is the converse true? That is, does the decimal expansion of every rational number eventually either recur or terminate.

ANSWER: That this is so may be proved by considering the process used in calculating the expansion

$$\frac{a}{b} = (.\, c_1c_2 \ldots c_m c_{m+1} \ldots c_n \ldots)_S .$$

At the "nth" stage we subtract $c_n \times b$ and leave a remainder r_n, such that $0 \leq r_n < b$. If any two remainders r_m and r_n are equal, it is clear that the block of digits $c_{m+1}c_{m+2}..c_n$.. will recur. But since there are only b-1 different non-zero remainders, after b stages of the process either a remainder of zero has occurred, in which case the expansion terminates, or else at least two remainders are equal, and the expansion recurs. The period (the number of digits in the recurring block) is plainly not greater than b-1 , irrespective of the scale used in the representation.

Q4. Given $\frac{a}{b}$ what can be said about the number of digits before the recurring block, and about the period, of the expansion.

ANSWER: Referring to the answer to Q2. if

$$\frac{a}{b} = (.c_1c_2 \ldots c_n \overline{d_1 \ldots . d_n})_S \quad \text{then} \quad \frac{a}{b} = \frac{A}{S^k(S^n-1)}$$

so that b is a factor of $S^k(S^n-1)$. It can be proved that k and n are always the smallest integers such that b divides $S^k(S^n-1)$. Thus for S = 10 , if $b = 2^l 5^m b_1$ where b_1 is relatively prime to 10 , then k is the larger of l and m, and n is the smallest power of 10 which leaves a remainder of 1 on division by b_1 .

For example, the decimal expansion of $\frac{23}{740} = \frac{23}{2^2 \times 5 \times 37}$ will have $k = 2$, and $n = 3$ (since 37 is a divisor of $999 = 10^3-1$, but not of 9 or 99).

In particular, if b is relatively prime to S, then $k = 0$, i.e., the decimal expansion is a "pure" recurring decimal.

Q5. Find all the fractions $\frac{a}{b}$ whose decimal expansion in the scale of 3 is a pure recurring decimal whose period is 4.

ANSWER: By the answer to Q4, b is a factor of 3^4-1, but not of $3-1$, 3^2-1, or 3^3-1. The possible values of b are 5, 10, 16, 20, 40 or 80. Then a is any integer prime to b. [If it is also given that b is a prime number, the only positive solutions less than one are

$$\frac{1}{5}, \frac{2}{5}, \frac{3}{5} \text{ and } \frac{4}{5}.]$$

The digits in the recurring blocks in decimal expansions often display curious properties.

Consider $\frac{1}{7} = (.\overline{142857})_{10}$. The remaining fractions with denominator 7 have expansions which may be obtained merely by cyclic rearrangement of this block of digits.

eg. $\frac{2}{7} = .\overline{285714}$. Again, breaking the block into halves and adding: - $142 + 857 = 999$; or into thirds and adding:- $14 + 28 + 57 = 99$; both yield suggestive results. The reader may find it interesting to discover further properties of a like nature, and to attempt to explain when and why they occur.

C. D. Cox.
