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UNSOLVED PROBLEMS 2.

THE HADAMARD PROBLEM .

The theory of combinatorial configurations abounds in unsolved problems, some of which can be stated in simple non-technical terms. One of the most famous among these is the following problem due to the French Mathematician, J. Hadamard.

Given an odd number of the form $n = 4m-1$, is it possible to select from among the subsets of the set of numbers $\{1,2, \ldots ,n\}$, n subsets $S_1 , S_2 , \ldots , S_n$ such that

(i) each S_i contains exactly $2m-1$ numbers, and

(ii) every pair of subsets S_i , S_j $(i \neq j)$ has exactly $m - 1$ numbers in common.

Hadamard formulated his problem slightly more technically, in terms of so called orthogonal matrices, but it can easily be shown to be equivalent to the combinatorial problem stated above.

Hadamard configurations (or, briefly, H. configurations) are known to exist for many values of m , and Hadamard conjectured that they exist for every m , but no one has yet been able to supply a general proof. A number of new H. configurations have recently been discovered with the help of fast electronic computers. At the present, the lowest value of m for which no one has yet been able to produce an H. configuration is $m = 29$.

For small values of m a little experimenting will readily yield H. configurations. In the case of $m = 1$, $n = 3$, we have the trivial configuration

$$S_1 = (1) \ , \ S_2 = (2) \ , \ S_3 = (3) \ .$$

In the case of $m = 2$ we are required to find 7 subsets containing 3 elements each so that every pair will have exactly one element in common.

$$S_1 = (123)\ ,\ S_2 = (145)\ ,\ S_3 = (167)\ ,$$
$$S_4 = (246)\ ,\ S_5 = (257)\ ,\ S_6 = (347)\ ,\ S_7 = (356)$$

is such a configuration.

It is easy to show that if there exists an H. configuration for $n = 4m - 1$ then there also exists one for $2n + 1 = 8m - 1$. If S is a subset of $M = \{1,2, \ldots, n\}$, let us denote by S' the complement of S in M, that is the subset of those elements of M which are not in S. For example, if $n = 7$ and $S = (145)$ then $S' = (2367)$. Clearly if S contains $2m - 1$ elements then S' contains $n - (2m - 1) = 2m$ elements.

Denote now by S'' the subset of $N = (1,2, .., n, n+1, .., 2n+1)$ obtained from S by adding n to each of its elements. For example, in the previous case when $S = (145)$, we have $S'' = (8,11,12)$. The reader should verify that

(a) If S_1 and S_2 are subsets of M, each containing $2m - 1$ elements and having $m - 1$ elements in common then S_1 and S_2' have m elements in common.

(b) If $S_1, \ldots, S_n$ is an H.configuration of M then the following $2n+1$ subsets of N form an H.configuration:

$$(S_i\ ,\ S_i''\ ,\ 2n + 1)\ ,\quad i = 1,2, \ldots, n$$
$$(S_i', S_i'')\ ,\quad i = 1,2, \ldots, n$$
$$M'' = \{n + 1, \ldots, 2n\}.$$

From (b) it follows that there exists an H.configuration for every $m = 2^k$, $k = 0,1,2, \ldots$. The first value of m not of this form is $m = 3$.

The reader should try to construct a Hadamard Configuration for $m = 3$, $n = 11$.

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