THE METHOD OF MATHEMATICAL INDUCTION.

The word induction is used to describe two quite different processes for reaching conclusions. One is the process by which a general result is hypothesised after observing its truth in a number of special cases. For example, one might arrive at the general proposition, "Green coloured fruit is unripe" as a result of several disappointing experiences with peaches, pears, oranges and so on. There is obviously always the possibility that statements arrived at in this way may not be universally valid. The above statement, for example, certainly is not true for avocado pears.

In common with people in all other walks of life, mathematicians make frequent use of this inductive process to arrive at conjectures or hypothesis, to suggest formulae and theorems. For example, after examining the following:-

$$1^3 = 1^2 = \left[\frac{1.2}{2}\right]^2$$

$$1^3 + 2^3 = 9 = (1 + 2)^2 = \left[\frac{2.3}{2}\right]^2$$

$$1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2 = \left[\frac{3.4}{2}\right]^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1 + 2 + 3 + 4)^2 = \left[\frac{4.5}{2}\right]^2$$

it is natural to guess that:-

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots. + n^3 = \left[\frac{n.(n + 1)}{2}\right]^2$$

for every positive integer n . (1)

Again, consider the values attained by the function $n^2 - n + 41$ for positive integer values of n . For $n = 1,2,3,4,5$ these are 41,43,47,53 and 61 respectively, all prime numbers. Arguing inductively, one might now be prepared to guess the following general result:-

"If n is a positive integer $n^2 - n + 41$ is a prime number." (2)

As it happens, the first of these statements is true, and the second is false; ((2) is true for all integers up to 40, but is obviously false for $n = 41$). It is clear that mathematical statements arrived at by this process of induction must be proved by some different method before their validity can be finally accepted.

The second use of the word induction is one which is special to mathematics. It is applied to a method by which statements like (1) above may actually be proved. Suppose $F(n)$ is such a statement, which is to be proved true for every positive integral value of n. The method of mathematical induction tells us that if we can prove $F(n)$ to be true for $n = 1$, and if also we can prove that its truth for $n = k$ implies its truth for $n = k + 1$, no matter what k may be, then we may deduce its truth for all n. For since it is true for $n = 1$, the second part of the method shows that it must be true for $n = 2$, and then for $n = 3$, and so on.

We shall prove the formula (1) by this method.

<u>Proposition:-</u> $1^3 + 2^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof

(i) Truth for $n = 1$:- $1^3 = \left[\frac{1.2}{2}\right]^2$.

(ii) Truth for $n = k$ implies truth for $n = k + 1$:-

For, if $1^3 + 2^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$

then

$$1^3 + 2^3 + .. + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2 + 4k + 4]$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 = \left[\frac{(k+1)(\overline{k+1}+1)}{2}\right]^2$$

<u>Q.E.D.</u>

Now here is an example that goes wrong. Why? (Answer p.32)

Theorem: In any set of n coloured objects, the objects all have the same colour.

Proof (i) Truth for $n = 1$: Obvious!

(ii) Truth for $n = k$ implies truth for $n = k + 1$.

1 2 3 .... k k + 1

0 0 0 0 0 0 0

k objects (1 to k)

k objects (2 to k + 1)

If we now know that in any set of k objects, the objects have the same colour, then if we are given a set of $(k + 1)$ objects labelled $1, 2, .., k + 1$; then we know that objects labelled $1, .., k$ all have the same colour, and objects labelled $2, .., k + 1$ all have the same colour.
Thus all the objects have the same colour. QED.

Now a second "wrong" example. Or is it wrong at all?

In a certain army camp, parades start at 6.30 a.m.. Parades are called A-parades if the privates cannot be sure at 9.a.m. on the previous evening that the parade will take place the following morning.

One Saturday, the commander announces that there will be an A-parade in the coming week (i.e., Sunday , Monday , ... , Saturday). The privates immediately start to wonder on which day the A-parade will fall. Certainly not on the Saturday, for if it had not occurred on or before Friday, they would be sure long before 9.p.m. on the Friday that it would have to occur on Saturday, the last day of the week - but then it would not be an A-parade. But an A-parade could not occur on Friday, because on Thursday, since the privates know that the parade could not occur on Saturday, it would have to occur on Friday. And so, by induction, they convince themselves that no A-parade can occur at all, and are very surprised when one does indeed occur on the Tuesday! Where did they err?

M. Arbib.

PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions and the names of those who send correct solutions by 31/12/65 will be published in the next issue of PARABOLA.

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior Section. All students may submit solutions of problems in the Open Section.

JUNIOR.

J31 As inquiries revealed that several students did not know the meaning of the phrase "relatively prime", we repeat the question.

Prove that, given five consecutive integers, it is always possible to find one which is relatively prime to all the rest.

Two integers are relatively prime (or, one is prime to the other) if they have no common factor greater than one.

(S. Byrnes has already submitted an excellent solution of this problem).