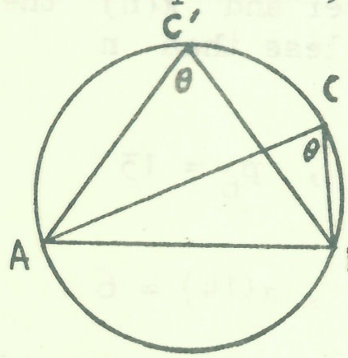
ANSWERS TO THE SENIOR COMPETITION QUESTIONS

Q.1 As for Junior Section - see Parabola Vol. 2 , No. 1.

Q.2 Three points A, B and C lie inside a circle whose radius is 1 inch. Show that $BC^2 + CA^2 + AB^2$ is less than or equal to 9 square inches.



ANSWER: It is easily seen that we may take A , B , C on the circumference of the circle.

We seek the position of the points in order that $AB^2 + BC^2 + CA^2$ is maximum. We first note that if $\theta < 90^\circ$ and $AC \neq BC$, then choosing C' so that AC' = BC' gives a larger value of the sum of squares. Since $\angle ACB = \angle AC'B$ the cosine law gives

$$AB^2 = AC^2 + BC^2 - 2AC.BC\cos\theta = AC'^2 + BC'^2 - 2AC'BC'\cos\theta$$

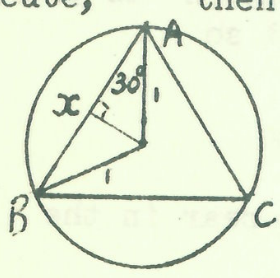
Since $\frac{1}{2}AC.BC = \frac{\text{Area } \Delta\ ABC}{\sin\theta} < \frac{\text{Area } \Delta\ ABC'}{\sin\theta} = \frac{1}{2}AC'.BC'$

we have $AB^2 + AC^2 + BC^2 = 2AB^2 + 2AC.BC.\cos\theta$

$$< 2AB^2 + 2AC'.BC'\cos\theta = AB^2 + AC'^2 + BC'^2$$

as asserted.

Thus if $AB^2 + BC^2 + CA^2$ is maximum where $\angle C$ and $\angle B$ are acute, then AC = BC ; and similarly AB = BC. The triangle is equilateral. If ABC is equilateral its side



$x = 2\cos 30^\circ = \sqrt{3}$ and

$AB^2 + BC^2 + AC^2 = 3 \times (\sqrt{3})^2 = 9$.

(Answer - Q.2 cont'd):

Since this is the maximum value we have $AB^2 + BC^2 + CA^2 \leq 9$ for all cases.

Q.3 Let p_n denote the n^{th} prime number and $\pi(n)$ the number of different prime numbers less than n.
For example -

$$p_1 = 2 \; , \; p_2 = 3 \; , \; p_3 = 5 \; , \; p_6 = 13$$

and

$$\pi(1) = 0 \; , \; \pi(2) = 0 \; , \; \pi(3) = 1 \; , \; \pi(14) = 6 \; .$$

Show that any positive integer N may be expressed in the form $\pi(n) + n$ or in the form $p_n + n$.

<u>ANSWER</u>: For small values of n

n	=	1	2	3	4	5	6	7 ..
$n + \pi(n)$	=	1	2	4	5	7	8	10 ..

and so we see that

$$(n+1) + \pi(n+1) = \begin{cases} \{n + \pi(n)\} + 1 & \text{if } n \text{ is not prime} \\ \{n + \pi(n)\} + 2 & \text{if } n \text{ is prime.} \end{cases}$$

Thus most numbers will be represented by this function. The ommitted numbers have the form $n + \pi(n) + 1$ when n is prime. But if $n = p_k$ is prime, then $\pi(n) = k - 1$ and so

$$n + \pi(n) + 1 = p_k + k - 1 + 1 = p_k + k \; .$$

Then the numbers omitted from the first series appear in the second.

Q.4 The polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

has the following properties:-

(i) All coefficients $a_0, a_1, \ldots, a_n$ are integers

(ii) For four different integers x, $f(x) = 0$.

Prove that if x is an integer then $f(x)$ is not a prime number. Construct a polynomial with integer coefficients which is zero for three different integer values of x and whose value is 5 for another integer value.

Can this polynomial take a prime number value other than 5 for an integer value of x ?

<u>ANSWER</u>: Let r_1, r_2, r_3 and r_4 be different integers such that $f(r_1) = f(r_2) = f(r_3) = f(r_4) = 0$.

By the remainder theorem,

$$f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)\, g(x)$$

where we see that $g(x)$ is also a polynomial with integer coefficients. Then if x is an integer each of the factors

$$x - r_1 \quad x - r_2 \quad x - r_3 \quad x - r_4 \text{ and } g(x)$$

is an integer. Since the r_i are all different, the first four of these integers are all different. Two could have values $+1$ and -1 but the other two are integers different from ± 1 and so these give two proper factors of $f(x)$. Thus $f(x)$ is not prime.

This shows how we can construct a polynomial which is zero for three different integer values for x and equal to five for another integer value of x . For example

$$-x(x - 2)(x + 4)(x^2 - x + 1)$$

has this property.

Q.5 The pairs of numbers sequence (4,4), (4,8), (6,10), (8,14) ... is formed according to the following rule. Representing the first pair by (x_1, y_1), the second by (x_2, y_2) etc., so that the nth pair is (x_n, y_n); the rule is:-

$$2x_{n+1} = x_n + y_n \quad \text{and} \quad 2y_{n+1} = 3x_n + y_n$$

determine $3x_n^2 - y_n^2$ for $n = 1, 2, \ldots$. What can you say about the value of y_n/x_n when n is very large?

ANSWER: We find that for

$$n = 1 \; , \; 2 \; , \; 3 \; , \; 4 \; , \ldots.$$

$$3x_n^2 - y_n^2 = 32 \; , -16 \; , \; 8 \; , -4 \; , \ldots.$$

This suggests that

$$3x_n^2 - y_n^2 = -64(-\tfrac{1}{2})^n$$

we can prove this by induction on n. It is true for $n = 1$. Suppose it is true for $n \geqq 1$, then for the case $n+1$ we have

$$3x_{n+1}^2 - y_{n+1}^2 = 3\left(\frac{x_n + y_n}{2}\right)^2 - \left(\frac{3x_n + y_n}{2}\right)^2$$

$$= -\tfrac{1}{2}(3x_n^2 - y_n^2)$$

$$= -\tfrac{1}{2} \cdot (-64)(-\tfrac{1}{2})^n \quad \text{by the induction hypothesis}$$

$$= -64 \cdot (-\tfrac{1}{2})^{n+1} \; ;$$

thus the result is proved.
When n gets large, $3x_n^2 - y_n^2$ becomes very small, but y_n becomes large. Therefore

$$3\left(\frac{x_n}{y_n}\right)^2 - 1 = \frac{3x_n^2 - y_n^2}{y_n^2}$$ becomes very small, so $3\left(\frac{x_n}{y_n}\right)^2 - 1$

is close to 0, i.e. $\frac{x_n}{y_n}$ is close to $\frac{1}{\sqrt{3}}$.

Q.6 $a_1, a_2, \ldots, a_n$ are n positive numbers.

If $k_1, k_2, \ldots, k_n$ is any arrangement of the integers $1, 2, \ldots, n$ there are $n!$ fractions

$$\frac{1}{a_{k_1}(a_{k_1}+a_{k_2})(a_{k_1}+a_{k_2}+a_{k_3})\ldots\ldots(a_{k_1}+a_{k_2}+\ldots+a_{k_n})}.$$

Show that their sum is

$$\frac{1}{a_1 a_2 \ldots\ldots a_n}.$$

ANSWER: We proceed by induction on n. The result is true if $n = 1$. Suppose $n \geqq 1$ and that the result is true for n. We shall show it holds for $n + 1$.

In each of the fractions, the last factor

$$(a_{k_1} + a_{k_2} + \ldots + a_{k_{n+1}}) = (a_1 + a_2 + \ldots + a_{n+1})$$

is always the same and so we can take it out as a common factor. Then for $m = 1, 2 \ldots n + 1$, we group together the fractions whose second last factor

$$(a_{k_1} + a_{k_2} + \ldots + a_{k_n}) = (a_1 + a_2 \ldots + a_{m-1} + a_{m+1} + \ldots a_{n+1})$$

Using the induction hypothesis, the sum of all the fractions in the mth group will be

$$\frac{1}{a_1 + a_2 + \ldots + a_{n+1}} \cdot \frac{1}{a_1 a_2 \ldots a_{m-1} a_{m+1} \ldots a_{n+1}}$$

$$= \frac{1}{a_1 + a_2 + \ldots + a_{n+1}} \cdot \frac{a_m}{a_1 a_2 \ldots a_{n+1}}$$

(Answer - Q.6 cont'd):

Then summing up these sums we get the final total

$$\frac{1}{a_1 + a_2 + \ldots + a_{n+1}} \quad \frac{a_1 + a_2 + \ldots a_{n+1}}{a_1 a_2 \ldots\ldots a_{n+1}} = \frac{1}{a_1 a_2 \ldots a_{n+1}} .$$

ANSWERS.

JUST FOLLOW DIRECTIONS - (p.26)

(1) ONE WORD .

(2) ~~S~~ B ~~I~~ A ~~X~~ ~~L~~ ~~E~~ N ~~T~~ A ~~T~~ ~~E~~ ~~R~~ N A ~~S~~

PUZZLE - (p.26)

$$\begin{array}{r} 17 \\ \times \quad 4 \\ \hline 68 \\ + \quad 25 \\ \hline 93 \\ \hline \end{array}$$

MATHEMATICAL INDUCTION - (p.12)

Part (ii) of the proof is not valid if $k = 1$.