SOLUTIONS OF PROBLEMS IN PARABOLA, VOL.2 , NO.2.

J31 This is set again in the Problem Section.

J32 A regular polygon is cut from a sheet of cardboard. A pin is put through the centre to serve as an axis about which the polygon can revolve. Find the least number of sides that the polygon can have in order that revolution through an angle $47\frac{1}{2}$ degrees will put it in co-incidence with its original position.

ANSWER: Suppose the polygon has n sides. The angle θ subtended by each of these sides at the centre O is $\frac{360^\circ}{n}$. Rotation about O leaves the polygon in co-incidence with its original position only if the angle of rotation ($47\frac{1}{2}^\circ$) is an integral multiple of θ.

Hence $47\frac{1}{2} \div \frac{360}{n} = \frac{9n}{144}$ is an integer.

The smallest n for which this is true is 144.

Correct solutions: S. Byrnes, Macquarie Boys High School;
K. McClymont, Vaucluse Boys High School.

J33 (i) Prove that if a and b are any two positive numbers

$$ab \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}.$$ When do equal signs hold.

(ii) Prove that if $a + b = 1$, where $a > 0$, $b > 0$,

then $$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

ANSWER J33:

(i) $0 \leq (a - b)^2 \leq 2(a - b)^2$ since no perfect square is negative, the equal signs applying only if $a = b$. Adding $4ab$ to each term gives, after simplifying

$$4ab \leq (a + b)^2 \leq 2a^2 + 2b^2 .$$

Now divide through by 4.

(ii) Applying the first inequality in (1), $ab \leq (\frac{1}{2})^2$;

hence $\frac{1}{ab} \geq 4.^{*}$

Again;

$$LHS = 2\left[\frac{(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2}{2}\right] \geq 2\left[\frac{(a + \frac{1}{a}) + (b + \frac{1}{b})}{2}\right]^2$$ (by the second inequality in (1)).

$$= \tfrac{1}{2}[(a + b) + (\tfrac{1}{a} + \tfrac{1}{b})]^2$$

$$= \tfrac{1}{2}[1 + \frac{a + b}{ab}]^2$$

$$= \tfrac{1}{2}[1 + {}^{1}/ab]^2$$

$$\geq \tfrac{1}{2}[1 + 4]^2 \quad \text{(using *)}$$

$$= \frac{25}{2} = RHS.$$

Correct : S. Byrnes, (part ii incomplete)
Partly
Correct : K. McClymont.

034 Three people, A, B and C, entered a competition. After the event, A reported "B was second, C was first". B said, "A was second, C was third". C said "A was first, B was third".

Each person's report contained one true statement and one false one. Which of A and B performed better in the competition?

ANSWER: If A's first statement is correct, B's first and C's second statements are both false. Their remaining statements are correct, showing that A was first, B second and C third.

If A's second statement is correct, B's second and C's first statements are false, and their remaining statements show that C was first, A was second and B was third.

In either case A performed better than B.

CORRECT SOLUTIONS: R. Morath (St. Aloys. Coll.); M. Polaska (O.L.M.C. Burraneer); A. Warner and P. Underwood (Manly H.S.); W.J. Müller (Newington Coll.); T. Osborn (Port Hacking H.S.); G. Thomas, P. Myers, J. McWhinney, P. Collins (Marist Bros. Eastwood); G. Lewis, J. Mock (S.B.H.S.); D. Hermann (Oakflats H.S.); H. Craft; M. McDonald (S.G.H.S); W.H. Wilson (Marsden H.S.); G. Onyetl (Balgowlah B.H.S.); G. Aitchison (Hurstville B.H.S.); D. Hales, (Canterbury B.H.S.); J. Hopping (Manly G.H.S.); J. Selby (Queenwood S); J. Walsh (De La Salle C., Ashfield); K. McClymont.

035 (i) A and B are disjoint sets (i.e., sets with no elements in common), each of which has cardinal number $\aleph_0$. Show that the set C which is the union of A and B (i.e., the set which contains all the elements in A and all the elements in B, and no other elements) also has cardinal number $\aleph_0$.

(ii) If $A_1, A_2, A_3, \ldots, A_n, \ldots$ are disjoint sets, each of which has cardinal number $\aleph_0$, prove that the union

035 - Question (ii) cont'd:

of all the sets A_n is a set whose cardinal number is $\aleph_0$.

ANSWER: (i) Let

$$\begin{array}{ccccccc} 1 , & 2 , & 3 , & \ldots\ldots\ldots , & n & , & \ldots. \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\ a_1 , & a_2 , & a_3 , & \ldots\ldots\ldots , & a_n & , & \ldots. \\ & & & & & & \\ b_1 , & b_2 , & b_3 , & \ldots\ldots\ldots , & b_n & , & \ldots. \end{array}$$

be (1-1) correspondences between the positive integers and the elements of the two sets A and B .

Then

$$\begin{array}{ccccccccccc} 1 , & 2 , & 3 , & 4 , & 5 , & 6 , & \ldots\ldots\ldots , & 2n-1 , & 2n , & \ldots. \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ a_1 , & b_1 , & a_2 , & b_2 , & a_3 , & b_3 , & \ldots\ldots\ldots , & a_n , & b_n , & \ldots. \end{array}$$

is a (1-1) correspondence between the elements of $A \cup B$ and the positive integers.

(ii)

$$\begin{array}{llccccccccccccl}
A_1 \equiv [& {}^1a_1 & \rightarrow & {}^2a_1 & & {}^3a_1 & \rightarrow & {}^4a_1 & & {}^5a_1 & \rightarrow \ldots & & {}^na_1 & \ldots] \\
 & & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & & & \\
A_2 \equiv [& {}^1a_2 & \leftarrow & {}^2a_2 & & {}^3a_2 & & {}^4a_2 & & {}^5a_2 & \ldots & & {}^na_2 & \ldots] \\
 & \downarrow & & & & \uparrow & & \downarrow & & \uparrow & & & & \\
A_3 \equiv [& {}^1a_3 & \rightarrow & {}^2a_3 & \rightarrow & {}^3a_3 & & {}^4a_3 & & {}^5a_3 & \ldots & & {}^na_3 & \ldots] \\
 & & & & & & & \downarrow & & \uparrow & & & & \\
A_4 \equiv [& {}^1a_4 & \leftarrow & {}^2a_4 & \leftarrow & {}^3a_4 & \leftarrow & {}^4a_4 & & {}^5a_4 & \ldots & & {}^na_4 & \ldots] \\
 & \downarrow & & & & & & & & \uparrow & & & & \\
A_5 \equiv [& {}^1a_5 & \rightarrow & {}^2a_5 & \rightarrow & {}^3a_5 & \rightarrow & {}^4a_5 & \rightarrow & {}^5a_5 & \ldots & & {}^na_5 & \ldots] \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
A_n \equiv [& {}^1a_n & & {}^2a_n & & {}^3a_n & & {}^4a_n & & {}^5a_n & \ldots & & {}^na_n & \ldots] \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots
\end{array}$$

035 - Answer (ii) cont'd:

The sets A_1, A_2, A_3 ... A_n, ... can be arranged as shown. The resulting array $\mathcal{a}$ can be put into a one-one correspondence with the integers by starting at ${}^1a_1$ and proceeding along the route shown. Thus

$$\begin{matrix} {}^1a_1 & {}^2a_1 & {}^2a_2 & {}^1a_2 & {}^1a_3 & {}^2a_3 & {}^3a_3 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{matrix}$$ and so on

The cardinal no. of $\mathcal{a}$ is $\aleph_0$.

Correct: W. H. Wilson.

Part (i) correct: W.J. Müller , S. Byrnes , G. Lewis.

Solutions of Part (ii) were correct only for a finite number of sets A_n .

036 If A is a set whose cardinal number is c , and B is a subset of A whose cardinal number is $\aleph_0$, show that the set (A-B) , consisting of the elements of A which are not contained in the subset B , has cardinal number c .

Deduce that the set of transcendental numbers has cardinal c .

ANSWER: Since the set of algebraic numbers (B) has cardinal number $\aleph_0$, and the set of all real numbers (A) has cardinal number c , the second statement is a particular case of the first. We shall content ourselves with proving this second statement. The proof can easily be modified to apply to the more general first statement.

We have to show that it is possible to set up a (1-1) correspondence between the set of transcendental numbers and the set of all real numbers.

036 Answer cont'd:

Select from the trancendentals the set C (of cardinal number $\aleph_0$) consisting of $\{\pi, 2\pi, 3\pi, \ldots n\pi, \ldots\}$. The union of this set with the set of algebraic numbers B , still has cardinal number $\aleph_0$, [by problem 035 (i)] i.e., it is possible to set up a (1-1) correspondence between the sets C and $B \cup C$. If such a (1-1) correspondence is set up and extended by letting any trancendental number not in C correspond to itself, we have a (1-1) correspondence between the transcendental numbers and the set of all real numbers.

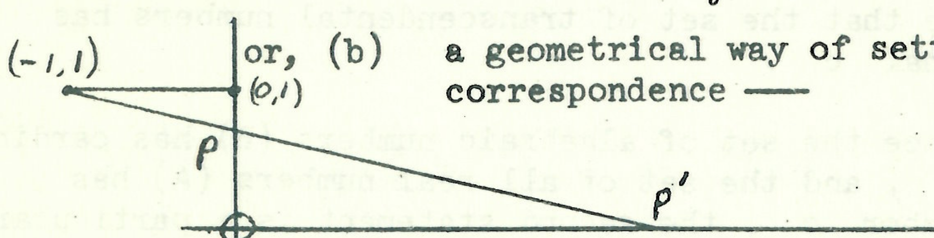
This was a very difficult question, and while there were a few interesting attempts, none was quite successful.

037 Show how to set up (1-1) correspondences between the points (o, y) , $o < y < 1$, on the y axis and

(i) the positive half of the x-axis
(ii) the entire x-axis
(iii) all points (x, y) in the unit square $o < x < 1$, $o < y < 1$.

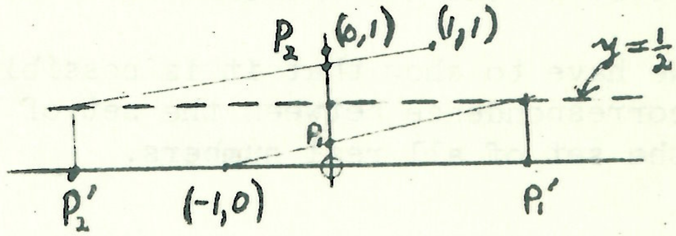
ANSWER: (i) For example, (a) $(o , y) \leftrightarrow (\frac{1}{y} - 1 , o)$;

or, (b) a geometrical way of setting up a correspondence ——



(ii) For example, $\left(o , y\right) \leftrightarrow \left(\frac{2y - 1}{y(1-y)} , o\right)$

or, a different geometrical example:-



037 Answer cont'd:

(iii) This is difficult. If $o < y < 1$ and the decimal expansion of y is

$$o.c_1\ c_2\ c_3\ \ldots\ c_{2n-1}\ c_{2n}\ \ldots$$

set up the correspondence

$$(o\ ,\ y) \rightarrow (o.c_1\ c_3\ c_5\ \ldots\ c_{2n-1}\ \ldots\ ,\ o.c_2\ c_4\ \ldots\ c_{2n}\ \ldots)$$

At first sight this appears to do the trick, but in fact it is not quite adequate. (Can you see why the correspondence is not (1-1)?). A slight modification corrects the argument, but to save space we shall omit this.

Correct solutions of Parts (i) and (ii) from W.H. Wilson, H. Craft and G. Lewis.

038 Prove that if all coefficients of the quadratic equation

$$ax^2 + bx + c = 0$$

are odd integers, then the roots of the equation cannot be rational.

ANSWER: (i) Suppose on the contrary that $x = {}^p/_q$ is a solution where p and q are relatively prime integers.

Then $ap^2 + bpq + cq^2 = 0$. But the left hand side is an odd integer, being the sum of 3 odd integers or of 1 odd integer and 2 even integers (p and q are not both even); it therefore cannot be equal to 0 . This contradiction establishes the result.

(ii) The discriminant is $\Delta = b^2 - 4ac$. The square of any odd number leaves a remainder of 1 on division by 8 . $\therefore$ $b^2 = 8k + 1$ for some k . The remaining term

038 - Answer (ii) cont'd:

$4ac$, is divisible by 4 but not by 8 and therefore leaves a remainder of 4 on division by 8 .

$$\therefore \quad 4ac = 8l + 4 \quad \text{for some } l \; .$$

$$\Delta = 8(k - l - 1) + 5 \; .$$

Hence Δ is not a perfect square, since its remainder on division by 8 is not 1 but 5. It follows that the equation cannot have rational roots.

CORRECT: D. Hales , A. Warner , P. Underwood , W.J. Muller, S. Byrnes , P. Myers , J. Mock , D. Hermann, P. Collins , H. Craft. Partly Right: J. Walsh , K. McClymont.

039 In a certain community, 1) any person who is married and plays tennis does not own a car, but 2) if a man plays tennis, he owns a car. 3) Any person who does not own a car is either single and a tennis player or female and not a tennis player. 4) All the car owners who are still single are female, and in fact, 5) there is no single female tennis player who does not own a car. What can one conclude about a person in the community who is known to play tennis?

ANSWER: By 1) and 2) no married man plays tennis.

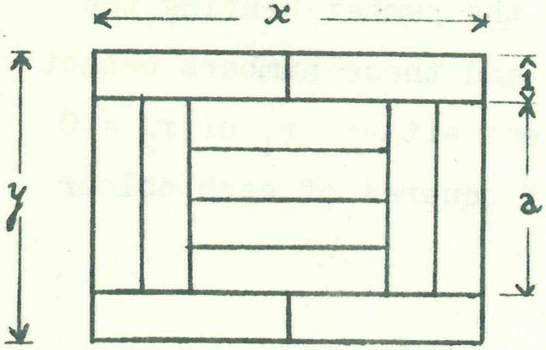
By 2) and 4) no single man plays tennis.

By 1) and 3) no married woman plays tennis.

Therefore all tennis players must be single and female, and hence also car owners by 5) .

CORRECT: A. Warner , P. Underwood , W.J. Muller , S. Byrnes , P. Myers , J. Hopping , J. Walsh , G. Lewis. Partly Right: H. Craft.

040.



The diagram shows a rectangular floor x units long and y units wide, which is covered with tiles 1 × a units.
(a , x , and y are all positive integers.) Prove that such a covering is possible (without cutting any tiles) if and only if "a" divides either x or y exactly.

ANSWER: The sufficiency of the condition is obvious, thus if a divides x , say, $x = ka$, the floor can be divided into k strips a by y units, each of which can be covered by a row of y tiles. To prove that the condition is necessary divide the floor into 1 × 1 squares, and colour the squares using "a" different colours, as shown in the diagram.

		a	1	2				a	1
3			a	1	2				a
2	3			a	1	2			
1	2	3			a	1	2		

Thus use the colour 1, for the lower left square, 2 for the two squares which have an edge in common with it, 3 for the next diagonal consisting of 3 squares, and so on up to the colour "a" . Then use "1" again and repeat the colours in the same order until every square is coloured. Whichever of the 2 possible ways a tile is laid, it covers "a" squares, one of each colour. Hence a necessary condition for the tiling to be possible is that there should be equal numbers of squares of each colour. There are certainly equal numbers of squares of each colour in the top a rows (since every column contains one square of each colour in these rows). We may omit these rows, and similarly, we may omit "a" columns along the right hand side of the figure. In fact, if

$$x = q_1 a + r_1 , \quad 0 \leq r_1 < a \quad \text{and} \quad y = q_2 a + r_2 , \quad 0 \leq r_2 < a$$

by omitting q_1 blocks of a rows, and then q_2 blocks of a columns we are left with a small rectangle in the L.H. bottom corner of the floor, with sides of length r_1 and r_2 , respectively, (unless one of these numbers is zero). But such a figure cannot have the same number of squares of each colour. In fact, the number of squares bearing the colour "a" is

040 Answer cont'd:

$r_1 + r_2 - a$, whilst, if $r_1 \geq r_2$, the number bearing the colour corresponding to r_1 is r_2 , and these numbers cannot be equal since $r_1 < a$. Hence, unless either r_1 or $r_2 = 0$ the floor cannot have equal numbers of squares of each colour and the tiling is impossible.

No successful attempts.

PUZZLE

```
                X  X
     Multiply      X
                ----
                X  X
     add        X  X
                ----
                X  X
                ====
```

Replace the crosses by digits using every digit except zero exactly once.

Answer p. 32.

JUST FOLLOW DIRECTIONS

(1) **Rearrange the letters in NEW DOOR to spell one word.**

(2) **Delete six letters from the following so that the remaining letters spell a word.**

SBIAXLENTATERNAS.

Answer p. 32.