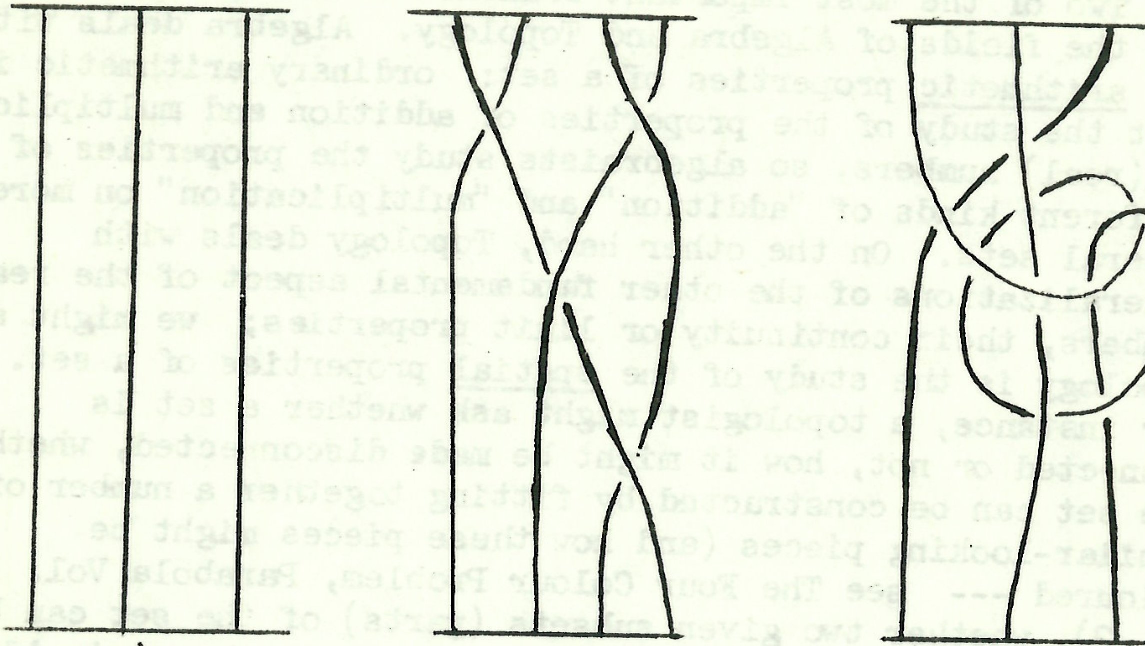
Parabola Volume 3, Issue 1 (1966)

BRAIDS

Two of the most important branches of modern mathematics are the fields of Algebra and Topology. Algebra deals with the arithmetic properties of a set: ordinary arithmetic is just the study of the properties of addition and multiplication of (real) numbers, so algebraists study the properties of different kinds of "addition" and "multiplication" on more general sets. On the other hand, Topology deals with generalizations of the other fundamental aspect of the real numbers, their continuity or limit properties; we might say Topology is the study of the spatial properties of a set. For instance, a topologist might ask whether a set is connected or not, how it might be made disconnected, whether the set can be constructed by fitting together a number of similar-looking pieces (and how these pieces might be coloured --- see The Four Colour Problem, Parabola Vol. I No. 2), whether two given subsets (parts) of the set can be pushed and pulled about within the set until they coincide. Topology has (too) often been called "Rubber-sheet Geometry", since it is concerned with those geometric properties of a set which are unchanged when the set is continuously deformed; this is inaccurate, Topology deals with much more than this, but the field is such a vast one now-a-days that a concise definition is impossible. Suffice to say that the word "continuous" is the key word in the previous sentence, and whereever the words "continuous" and "limit" are used in mathematics, there you will find Topology.

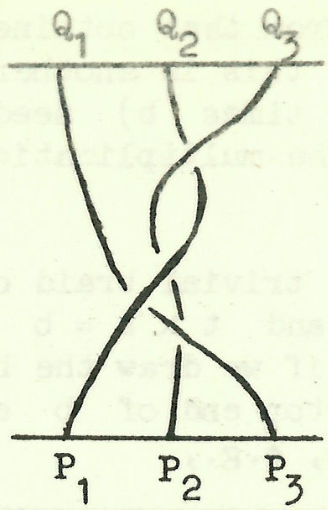
The pretty little topic of Braid Theory combines Algebra with Topology. A braid consists of a number of strings in space with their ends held fixed on two parallel lines (so that the whole thing won't come undone), and such that each string proceeds from its starting point to its end point without doubling back on itself (the strings can't be knotted). Thus diagrams 1) and 2) are braids, 3) is not. Of course in our mathematical model, the strands of a braid will be curves; in the everyday world, however, the strands can be anything from string (in which case the braid is sometimes called "rope") to human hair.

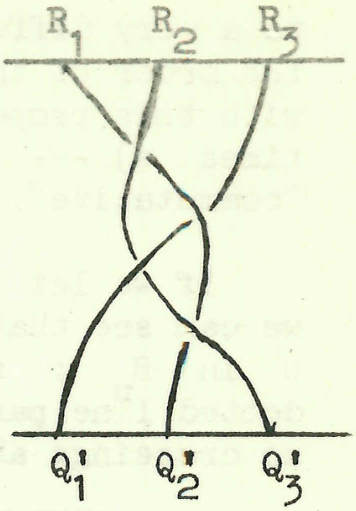


1) 2) 3)

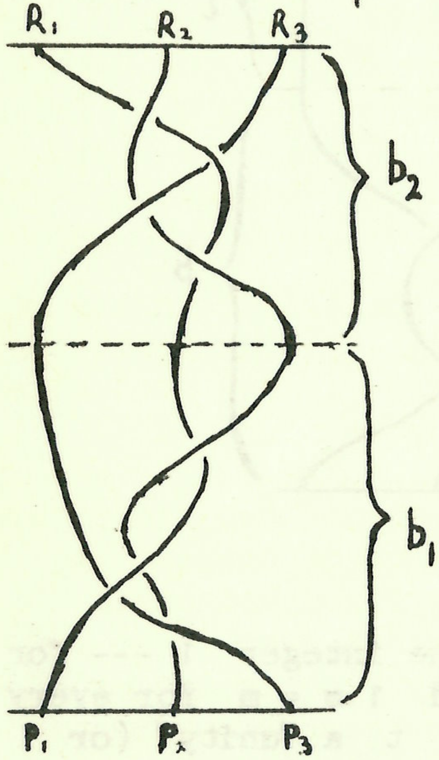
The braid of diagram 1) is called the trivial braid on 3 strings. There is of course a similar trivial braid on n strings, for each positive integer n . Let us denote the set of all braids on n strings by B_n .

Braids are topological concepts (because two braids are considered to be identical if the strings of one can be moved about, always holding their ends fixed and never allowing them to double back on themselves, until the two braids coincide); now for the algebra. Two braids on n strings can be "multiplied" together by simply making one braid after the other; more precisely, if b_1 is a braid with starting points $P_1, P_2, \ldots P_n$, and end points $Q_1, Q_2, \ldots Q_n$, and b_2 is another braid on n strings, we say that the "product" $b_1 \times b_2$ is the braid formed by releasing the fixed points $Q_1, \ldots Q_n$ and regarding them as the starting points of the braid b_2 . For example, if

b_1 is the braid 

and b_2 is 

then the product $b_1 \times b_2$ is the braid



which can be simplified to

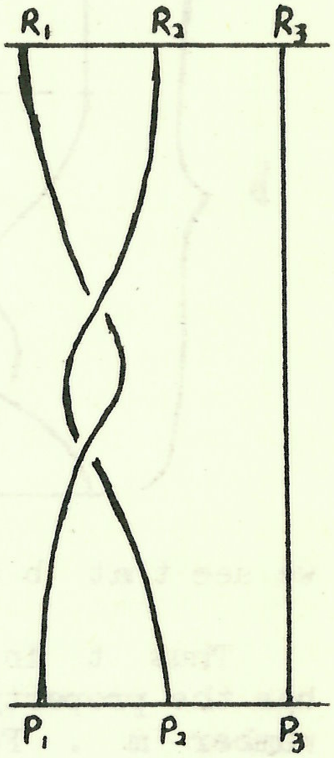
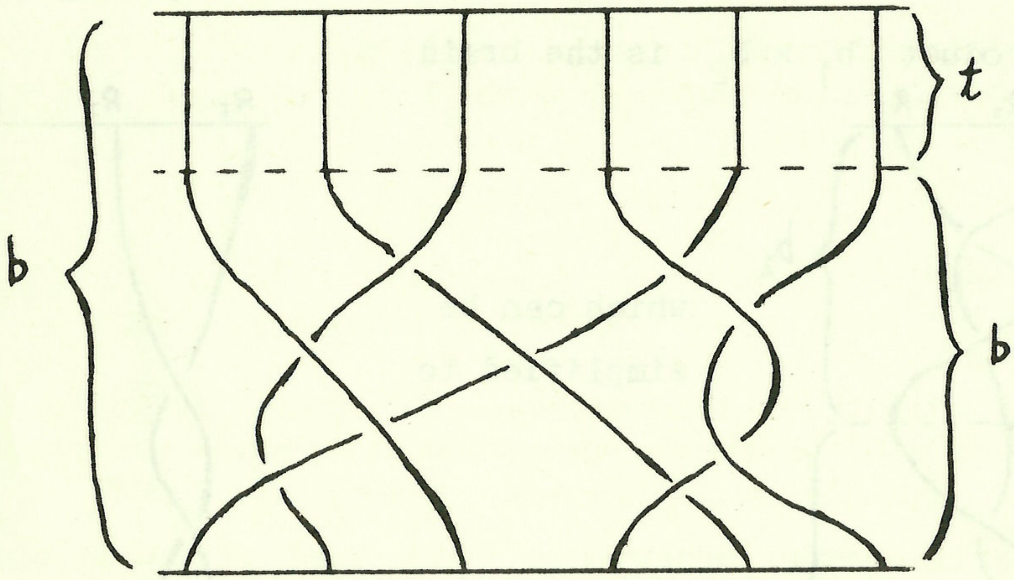


Diagram 4)

This "multiplication" has some rather peculiar properties, as we shall see. In particular, we must be careful about the order in which the multiplication is made, for it may happen that $b_1 \times b_2$ is not the same as $b_2 \times b_1$! (Can you find a pair of braids b_1 , b_2 such that $b_1 \times b_2 \neq b_2 \times b_1$?)

(While we're on the subject of generalized multiplications, we should note that this phenomenon often happens. It's clear that putting on your socks and then putting on your shoes leads to a very different result from that obtained by reversing the order of the operation; this is another "multiplication" with this property that (a times b) need not equal (b times a) --- we say that the multiplication is not "commutative".)

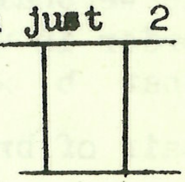
If we let t denote the trivial braid on n strings, then we can see that $b \times t = b$ and $t \times b = b$ for every braid b in B_n ; for instance, if we draw the braid b and put a dotted line parallel to the top end of b so that there are no crossings above this line, e.g.,



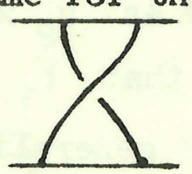
we see that $b = b \times t$.

Thus t in B_n is rather like the integer 1 --- for 1 has the property that $m.1 = m$ and $1.m = m$ for every number m . For this reason we call t a "unity" (or "identity") in B_n .

Let's have a closer look at B_2 , the braids on just 2 strings. We have a name for the trivial braid t

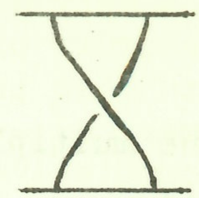


let's call the braid

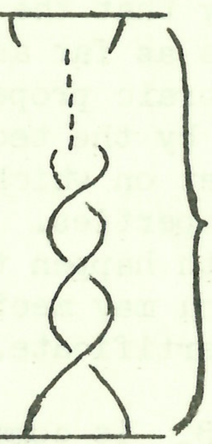


s . Notice that

the braid s' formed by passing the 1st string under the 2nd, instead of over as in s, has the property that $s \times s' = t$ and $s' \times s = t$ --- in a sense s' undoes s and s undoes s'. If t is analogous to 1, then s' is analogous to $1/s$ or s^{-1}. Thus instead of s' we shall use the symbol s^{-1} (read "s inverse" or "the inverse of s") for this braid.



Now every braid b on 2 strings is a product of the braids s or s^{-1}, where $b = s \times s \times s \times \ldots \times s$ (m times) if b is the braid



or $b = s^{-1} \times s^{-1} \times \ldots \times s^{-1}$ if b has m twists in the opposite direction. We write $s \times s \times \ldots \times s$ (m times) in shorthand as s^m, and $s^{-1} \times s^{-1} \times \ldots \times s^{-1}$ as s^{-m}. This turns out to be more than just convenient notation, for these symbols behave just like the powers of s and s^{-1}.

Thus $s^m \times s^n = s^{m+n}$ for any integers m, n positive or negative (where $s^1 = s$ and s^0 means the trivial braid t)!

(At first sight this might appear obvious, but remember that s and s^{-1} are not ordinary numbers, and we are not talking about ordinary multiplication, so this statement needs to be proved by looking at the braids involved.)

We can set up a 1-1 correspondence (see Vol.2, No.2) between B_2 and the set of positive and negative integers by

$$t = s^0 \leftrightarrow 0$$
$$s^m \leftrightarrow m$$
$$s^{-m} \leftrightarrow -m \text{ . Then } s^{m+n} = s^m \times s^n \leftrightarrow m+n \text{ ,}$$

so the multiplication $\times$ in B_2 has been transformed into addition $+$ in the integers (c.f. logarithms: $\log ab = \log a + \log b$, so multiplication of (positive) real numbers is transformed into addition by the correspondence $a \leftrightarrow \log a$; here also the "unity" 1 for multiplication is made to correspond with $\log 1 = 0$, which is a "unity" for addition since $r + 0 = r$ for every number r). Because of this correspondence, the algebraist would say that the group of braids B_2 is the same as the group of integers as far as algebra is concerned, i.e., they have the same algebraic properties; we shall not say precisely what is meant by the technical word "group" at this stage, it is merely a set on which is defined a multiplication with some particular properties. Multiplication of braids and addition of integers both happen to have these "particular properties". Some of you may meet groups in the new syllabus for the Higher School Certificate.

Multiplication in B_2 is commutative, i.e. $b_1 \times b_2 = b_2 \times b_1$ for every pair of braids b_1, b_2 of B_2 . The situation changes drastically however when we look at braids of 3 or more strings.

In B_n , the set of braids on n strings, we distinguish a special set of $n - 1$ braids as follows:

σ_1 is the braid formed by passing the 1st string <u>over</u> the 2nd, leaving the 3rd to nth strings alone;

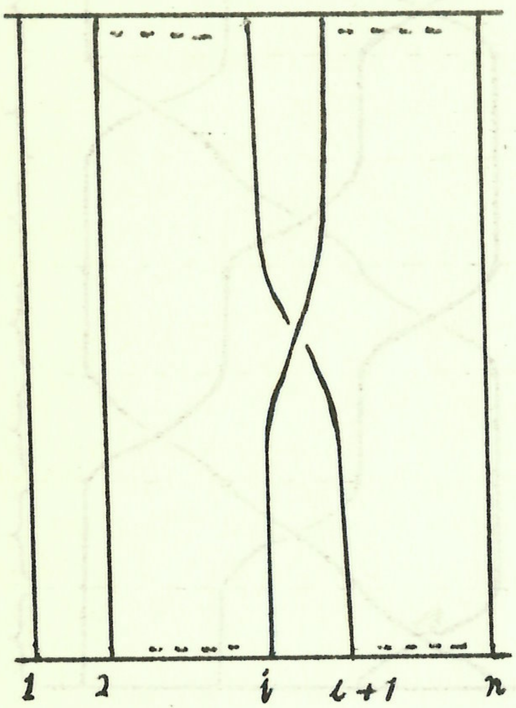
⋮

σ_i is formed by passing the i-th string over the $(i + 1)$st;

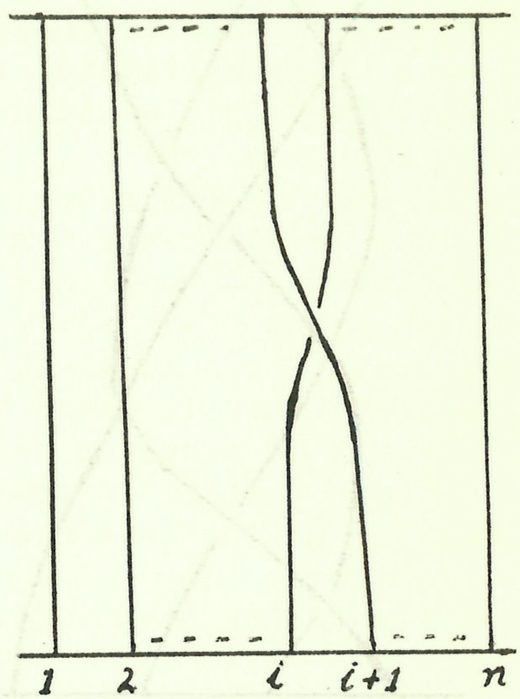
⋮

σ_{n-1} is formed by passing the (n-1)st string over the n-th.

In addition to these braids, there are their <u>inverses</u> σ_1^{-1}, ... σ_{n-1}^{-1}, where σ_i^{-1} is the braid formed by passing the i-th string <u>under</u> the (i-1)st.



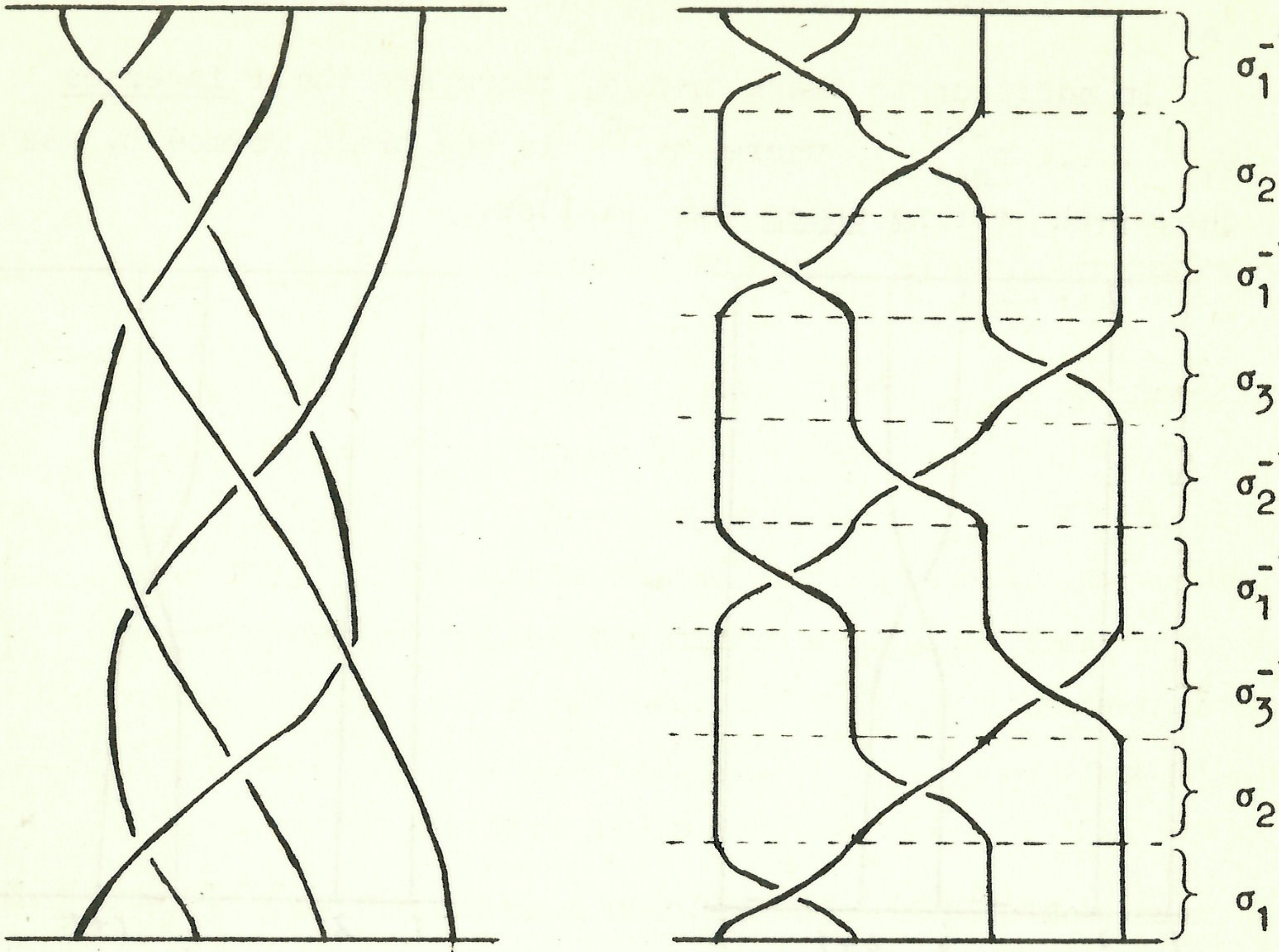
σ_i



σ_i^{-1}

Then $\sigma_i \times \sigma_i^{-1} = t = \sigma_i^{-1} \times \sigma_i$.

Clearly any braid on n strings can be broken up into a product of the braids σ_1, ... , σ_n and their inverses. For instance, consider the braid of the following diagram, with n = 4 . It can be written as a product in the following way: draw lines across the braid so that between any two lines exactly one crossing of the braid occurs e.g.,



Thus the braid is the product $\sigma_1 \times \sigma_2 \times \sigma_3^{-1} \times \sigma_1^{-1} \times \sigma_2^{-1} \times \sigma_3 \times \sigma_1^{-1} \times \sigma_2 \times \sigma_1^{-1}$ Unfortunately, a braid can be written as such a product in more than one way. You can convince yourself that in B_4 , the braid $\sigma_1 \times \sigma_3$ can also be regarded as the product $\sigma_3 \times \sigma_1$ also that $\sigma_1 \times \sigma_2 \times \sigma_1 = \sigma_2 \times \sigma_1 \times \sigma_2$ and $\sigma_2 \times \sigma_3 \times \sigma_2 = \sigma_3 \times \sigma_2 \times \sigma_3$. Another way of looking at this is to think of these as different ways of writing the trivial braid as a product; for instance, multiply both sides of $\sigma_1 \times \sigma_2 \times \sigma_1 = \sigma_2 \times \sigma_1 \times \sigma_2$ by σ_2^{-1} <u>on the right-hand side</u> (remember we must be careful of the order of multiplication) getting

$$\sigma_1 \times \sigma_2 \times \sigma_1 \times \sigma_2^{-1} = \sigma_2 \times \sigma_1 \times (\sigma_2 \times \sigma_2^{-1}) = \sigma_2 \times \sigma_1 \times t = \sigma_2 \times \sigma_1 \ .$$

Similarly multiply on the right by σ_1^{-1} and then by σ_2^{-1}, getting

$$\sigma_1 \times \sigma_2 \times \sigma_1 \times \sigma_2^{-1} \times \sigma_1^{-1} \times \sigma_2^{-2} = t .$$

(You should convince yourself that this equation is true, by constructing the braid on the left. In the general case of B_n, we get equations

$$\sigma_i \times \sigma_j = \sigma_j \times \sigma_i \quad \text{if } j \neq i-1 \text{ or } i+1 ,$$

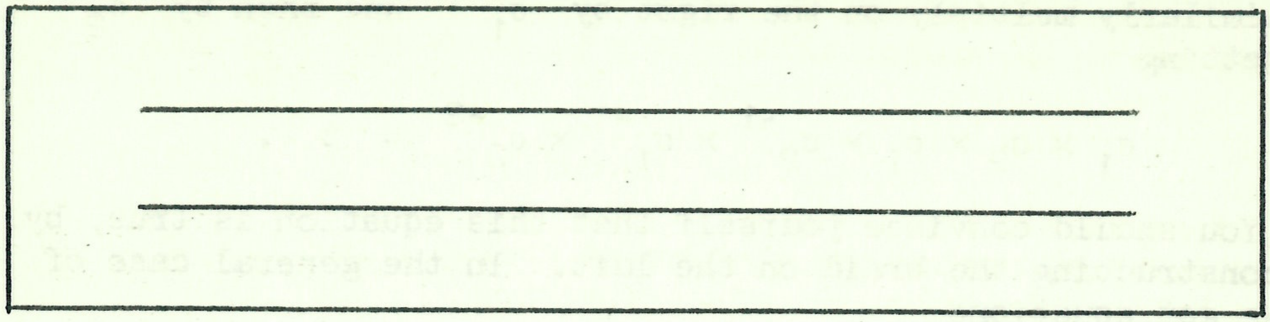
and

$$\sigma_i \times \sigma_{i+1} \times \sigma_i = \sigma_{i+1} \times \sigma_i \times \sigma_{i+1} \quad \text{for each } i \leq n-1.$$

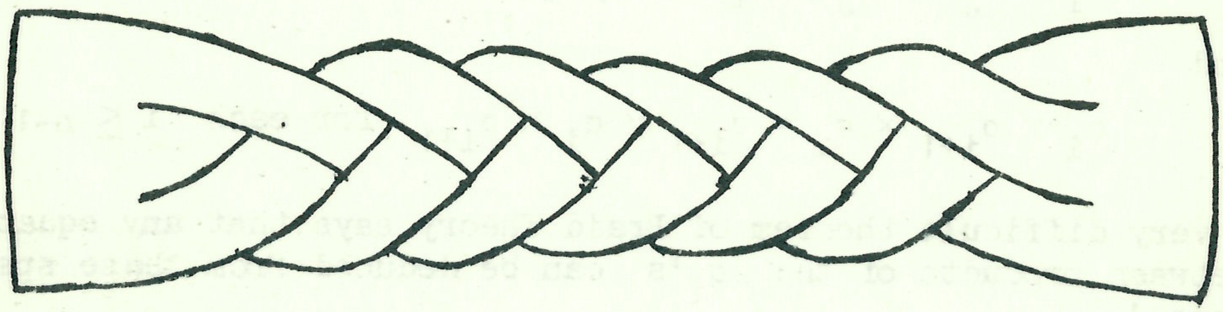
A very difficult theorem of Braid Theory says that any equation between products of the σ_i's can be deduced from these special ones.)

At this point, I was about to say that, while Braid Theory was quite interesting from the pure mathematician's point of view, I knew of no non-trivial applications. However, my appreciation of the subject gained immensely from a discussion with some teachers attending the Summer School for Mathematics Teachers held at this university in January; moreover I now consider Braid Theory to be a perfect answer to those people who say, "Oh yes, modern mathematics is all very pretty, with its sets and groups and all these abstract generalizations, but what does it do that the old-fashioned mathematics of geometry and calculus can't do?" In Braid Theory we run across an interesting problem, with quite important applications to industry, a solution for which involves the modern notions of topology and groups in a natural and non-avoidable way.

It seems that leatherworkers have known the following trick for some time. Take a strip of leather, and cut two straight slits as shown. The strip can then be plaited into the



following braid, without cutting the leather again!

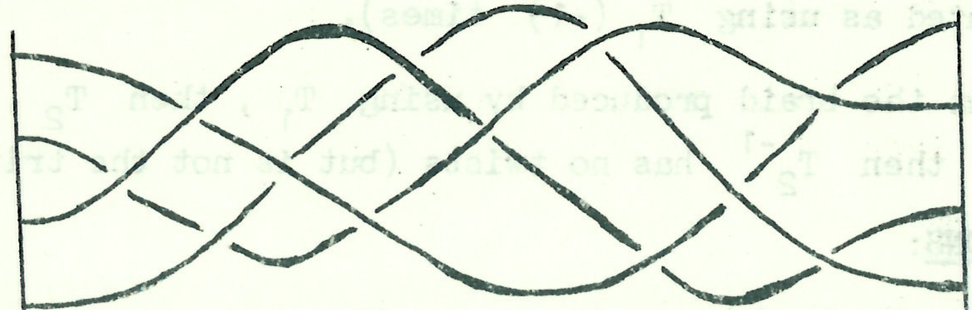


This looks very surprising at first, but a little thought shows how it is done. The ends of the strip are passed between the slits again and again in the appropriate order.

This is of interest to weavers, for it means that threads can be braided with one set of ends held fixed on stationary bobbins supplying the thread, the other set of ends fixed on a single shuttle on which the braid is wound, the shuttle moving between the threads; for some purposes such a weaving machine is easier to construct and maintain than one with a number of shuttles weaving a dance amongst the threads.

The problem is, exactly which braids can be made in this way from the trivial braid? Of course this method of construction cannot change the order of the strings on the ends (except possibly to reverse them). The trivial braid joins the i-th starting point to the i-th end point, for each i ; a braid which does this will be called principal. Thus the braid of diag. 2) is not principal, because it joins the 1st starting point to the 2nd end point; the braid of diag. 4) is principal. Clearly then we can only expect to solve the problem for principal braids.

An English mathematician solved the problem (in 1961 --- this really is modern mathematics). He showed that <u>any</u> principal braid on 3 strings can be so made, while there are certain braids on 4 and more strings which cannot be. (I imagine quite an amusing pastime could be made by fixing 3 strings to two small pieces of wood to form a trivial braid, and setting someone the problem of making a particular braid which has been drawn or made beforehand --- some braids are quite difficult. The problem could be varied by adding a fourth string and setting an impossible braid, such as the square sennit:



I am particularly indebted to one of the teachers, Mrs. Aitchison of Port Hacking, for the reference to this paper, as well as to the "Scientific American" issues of January and February, 1962, which refer to the leather braid, and also for a braid constructed of 3 strips of flat plastic to demonstrate this weaving technique.

The model poses another problem, which as far as I know is not discussed in the literature. When the ends of the braid are passed between the "strings", the flat strips of plastic (or leather) become twisted (this would not concern the weavers of wool, of course). But in some braids, when completed, for instance the leather braid drawn above, these twists cancel each other out; the question is, exactly which braids have this "flatness" property?

For 3 strings, the answer is as follows (unfortunately the proof requires some rather difficult results in group theory, so cannot be presented here): let T_1 be the operation of passing one end of the braid (the "shuttle") between the 1st and 2nd strings from above, and T_2 the operation of passing between the 2nd and 3rd strings from above.

Further let T_1^{-1} and T_2^{-1} be the "inverse" operations i.e., passing the end through from below. Then the twists in the final braid produced will all cancel if, and only if, the total algebraic number of times T_1 is used is 0, and the total number of times T_2 is used is 0, (where using T_1^{-1} is counted as using T_1 (-1) times).

Thus the braid produced by using T_1, then T_2, then T_1^{-1}, then T_2^{-1} has no twists (but is not the trivial braid).

QUESTIONS:

1. Let b be a braid in B_n. Show that there is a braid b' in B_n which undoes b, i.e., such that $b \times b' = t = b' \times b$. Such a braid may be called an inverse to b, and written b^{-1}.

2. Show that t is the only braid which is a unity, i.e., such that $b \times t = b = t \times b$ for every braid b. Also show that b has only one inverse braid b^{-1}.

3. If b is the product $\sigma \times \tau$ of braids σ and τ, how is b^{-1} related to the braids σ^{-1} and τ^{-1}? Give an example to show that $b^{-1} \neq \sigma^{-1} \times \tau^{-1}$ in general

4. Find two different braids σ and τ on 3 strings such that $\sigma \times \sigma = \tau \times \tau \times \tau$. Can you find two such braids which generate B_3 that is, with the extra property that every braid in B_3 can be written as a product of the

Question 4 cont'd:

braids σ, τ and their inverses σ^{-1} and τ^{-1}. (The braids σ_1 and σ_2 defined in the article generate B_3, but do not satisfy the additional property.)

5. If b is any braid, show that there is a positive (non-zero) integer n such that $b^n = b \times b \times \ldots \times b$ (n times) is principal. The smallest such integer n will be called the period of b. (A principal braid has period 1.) Which integers occur as the periods of braids on 2 strings? 3 strings? 4 strings?

6. (A harder one) - It is clear that if b_1 and b_2 are principal braids, then so are the braids $b_1 \times b_2$ and b_1^{-1}. If b has period p, and b^q is also a principal braid, show that q is a multiple of p.

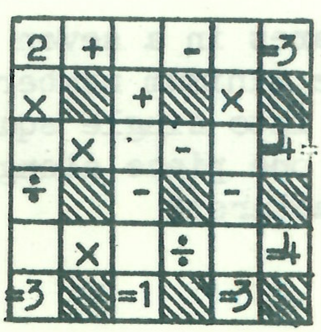
7. In terms of braids σ_1 and σ_2, what is the braid produced by the operation T_1 on the trivial braid? by T_2? by their inverses?

8. Show how the leather braid shown can be produced from the trivial braid using the operations T_1 and T_2 and their inverses.

   Check that your answer satisfies the "flatness" criterion of the last paragraph.

Dr. N. Smythe.

CROSS NUMBER PUZZLE.



There are no clues; just fill in the missing numbers, working from left to right and top to bottom, taking each symbol as it comes. Single figures only. No noughts.

(Solution Page 32).

EDITORIAL

The author of the article on Braids in this issue, Dr. N. Smythe, has recently returned from earning his doctorate of philosophy at Princeton University, U.S.A., where he specialised in the branch of topology known as Knot Theory.

The study of knots by inductive methods was begun fairly early in the historical development of topology (1850), but it remained something of a backwater until quite recently when the use of more sophisticated machinery showed how problems in know theory were related to some of the unsolved fundamental questions in general topology.

Although the details have not yet been finalised, it is expected that the annual School Mathematics Competition will be held, as usual, in May, with the customary generous donation of prizes by IBM (Australia) Pty. Ltd. If you enjoy tackling difficult problems which require insight and ingenuity to solve, ask the Principal of your school for an application form, and for details of the competition.

TWO OF A KIND

1) There are 87 entrants in a tennis tournament conducted in the usual knock-out fashion. How many matches are played altogether in all rounds?

2) A slab of chocolate consists of 28 squares in a seven by four rectangular array. What is the minimum number of operations required to divide it up into single squares if each operation consists in breaking one piece along one of the lines provided by the manufacturer?