SOLUTIONS OF PROBLEMS IN PARABOLA, VOL.2 , NO.3.

J31 Prove that, given five consecutive integers, it is always possible to find one which is relatively prime to all the rest.

Two integers are relatively prime (or, one is prime to the other) if they have no common factor greater than one.

ANSWER: At least two of the numbers are odd, and of any two consecutive odd numbers one at least is not divisible by 3 (since otherwise their difference, 2 , would also be divisible by 3, which is false). Thus it is always possible to find one of the numbers, x say, which is divisible by neither 2 nor 3 . This number is relatively prime to all the others since if y is any one of the others, a prime factor of x and y is also a factor of $|x - y|$, a positive integer not greater than 4 , which obviously has no prime factors other than 2 or 3 . Since neither of these divides x , there is no factor common to x and y .

Correct: S. Byrnes (Macquarie B.H.S.)

J41 In the following equation:-

$$29 + 38 + 10 + 4 + 5 + 6 + 7 = 99$$

the left hand side contains every digit exactly once. Either find a similar expression (involving only + signs) whose sum is 100 , or prove that it is impossible to do so.

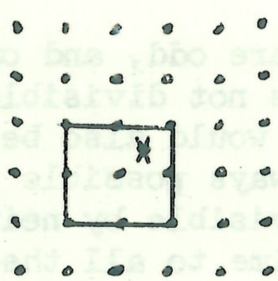
ANSWER: It is impossible to do so.
Note that $10a + b=[9a + (a + b)]$ leaves the same remainder on division by 9 as $(a + b)$. It follows that the left hand side leaves the same remainder on division by 9 as

J41 Answer cont'd:

$0 + 1 + 2 + 3 + \ldots + 9 = 45$, viz. 0. Since 100 is not a multiple of 9, it cannot be so represented.

Correct: P. Bardsley (Griffith H.S.); S. Byrnes.

J42 (i) X is a point at the centre of a square array of dots with $2n$ dots in each side. (In the diagram shown, $n = 3$). The diagram shows a square whose sides lie in the rows or columns of dots, and which encloses the point X. How many such squares may be drawn if $n = 1, 2, 3, \ldots, n$?



(ii) How many rectangles enclosing X may be drawn through the dots?

ANSWER: (i) The number of squares, $N(n)$, is $1^2 + 2^2 + 3^2 + \ldots + n^2 + (n-1)^2 + \ldots + 3^2 + 2^2 + 1^2$. Note that $N(1) = 1$ (obvious), $N(2) = 6$, $N(3) = 19$ and so on. The terms in the formula for $N(n)$ are respectively the number of squares enclosing X of side $1, 2, 3, \ldots, 2n-1$ units where 1 unit is the distance between a pair of neighbouring dots.

A neater formula for $N(n)$ is $\dfrac{n.(2n^2 + 1)}{3}$. To prove that

$$n(2n^2 + 1) = 3[1^2 + 2^2 + 3^2 + \ldots + (n-1) + n^2 + (n-1) + \ldots$$
$$\ldots + 2^2 + 1^2]$$

use mathematical induction, as follows:

(a) The formula is obviously true if $n = 1$;

(b) Suppose it is true when $n = k$, and consider $n = k+1$.

J42 (i) Answer cont'd:

$$\text{L.H.S.} = (k+1)[2(k+1)^2+1] = k(2k^2 + 1) + (6k^2 + 6k + 3)$$
$$= 3[1^2+2^2+ \ldots + k^2+(k\ 1)^2+ \ldots + 1^2] + 3[k^2+(k+1)^2]$$
$$= \text{R.H.S.}$$

Thus truth for $n = k$ implies truth for $n = k+1$ and the formula is true for all values of n.

(ii) The number of rectangles (including squares) which enclose X is n^4. There are clearly n ways of choosing the lines of dots containing each of the four sides, and since they can be chosen quite independently the total number of choices is found by multiplying, giving $n \times n \times n \times n$, the stated result.

Partly correct: S. Byrnes.

J43 We call $1 - x$ the complement of x, and, if $x \neq 0$, we call $1/x$ the reciprocal of x.

Starting with the number $\frac{1}{3}$, we form its reciprocal, 3, and its complement, $\frac{2}{3}$, and we find as many numbers as possible by repeatedly taking reciprocals or complements of numbers we have found so far. Which numbers are obtained by this process? Will the number of numbers obtained be different if we start with a different number (other than 0 or 1)?

ANSWER: Starting with $\frac{1}{3}$ we obtain $\frac{1}{3}, 3, \frac{2}{3}, -2, -\frac{1}{2}, \frac{3}{2}$; 6 numbers in all. In fact starting with x we obtain

$$x, \frac{1}{x}, (1-x), \frac{1}{1-x}, -\frac{(1-x)}{x}, \text{ and } -\frac{x}{1-x}.$$

J43 Answer cont'd:

Finding the reciprocal or the complement of any of these expressions always yields another of them. It follows that no more than 6 numbers can ever be obtained, whatever value of x we start with. Fewer than 6 numbers will be obtained for values of x for which two (or more) expressions are equal. Equating pairs of expressions yields the following permissible values of x :-

$$2\ ,\ \frac{1}{2}\ ,\ -1 \quad\text{and}\quad \frac{1}{2}(1 \pm \sqrt{-3})\ .$$

Starting with any of the first three yields this set of three numbers. We will ignore the last two numbers which are not "real" since the square root of a negative number is involved.

Correct: G. Aitchison (Hurstville B.H.S.) ; S. Byrnes.

O44 Prove that if the sum of the fractions $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ (where n is a positive integer) is put into decimal form in the scale of ten, the resulting decimal does not terminate.

ANSWER:

$$\frac{1}{n} + \frac{1}{(n+1)} + \frac{1}{(n+2)} = \frac{(n+1)(n+2) + n(n+2) + n(n+1)}{n(n+1)(n+2)}$$

Exactly one of n, $(n+1)$, and $(n+2)$ is divisible by 3. It follows that the denominator but not the numerator of the R.H.S. is divisible by 3, and this will continue to be the case when this fraction is reduced to lowest terms. However, the decimal expansion of the fraction a/b (in lowest terms) terminates only if the only prime factors of b are 2 and 5 (see Parabola, Vol.2, No.3 p.4).

Correct: S. Byrnes, D. Hermann (Oak Flats H.S.), L. Knight (Fairfield B.H.S.), G. Matheson (Drummoyne H.S.); P. Underwood, W.H. Wilson (Marsden H.S.).

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045 (i) The period (recurring block of digits) of a recurring decimal is of length n and begins at the $k + 1^{st}$ place to the right of the decimal point. Let

$$\frac{A}{10^k}$$

be the part preceding the period, and let P, an integer of n digits, be the period (e.g., in $3.6\overline{87}$, $k = 1$, $n = 2$, $A = 36$, $P = 87$). Prove that

$$A(10^n - 1) + P$$

is not divisible by 10.

(ii) Let $\frac{m}{2^\alpha 5^\beta q}$ $(\alpha \geq 0 , \beta \geq 0)$

be a fraction such that q is not divisible by 5 or 2, and let k be the greater of the two integers α, β. Using (i), prove that in the decimal expansion of this fraction the period begins at the $k + 1^{st}$ place to the right of the decimal point.

ANSWER: (i) $A(10^n - 1) + P$ is divisible by 10 if, and only if 10 divides $A-P$, i.e. if and only if the integers A and P end with the same digit, which is not true (since then the recurring block of digits would begin one digit further to the left).

(ii) [It must also be true that $\frac{m}{2^\alpha 5^\beta q}$ is in lowest terms.]

If $M = 2^{k-\alpha} 5^{k-\beta} m$ we have

$$x = \frac{m}{2^\alpha 5^\beta q} = \frac{M}{10^k q}$$ where M is not divisible

by 10. Let the decimal expansion begin in the $K + 1^{st}$ place

045 (ii) Answer cont'd:

and let A and P be as defined in part (i) of the question. Then $10^K x = A + .\bar{P} = A + \frac{P}{(10^n - 1)}$

$$x = \frac{M}{10^k q} = \frac{[(10^n-1)A + P]}{10^K(10^n - 1)}$$

$$\therefore \qquad M\,10^K(10^n-1) = [(10^n-1)A + P]\,10^k q \quad .$$

Since neither M nor $[(10^n-1)A + P]$ are divisible by 10 while q and 10^n-1 are each prime to 10, the highest powers of 10 which divide the two sides are 10^K and 10^k. Hence $K = k$ as was to be proved.

Partly Right: S. Byrnes.

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046 Carry out the division processes for finding the decimal expansions of

$$\frac{a}{41} \text{ for } a = 1, 2, \ldots, 40,$$

and list the distinct remainders in each case. The table you will obtain begins as follows:

a	remainders
1	1, 10, 16, 18, 37;
2	2, 20, 32, 33, 36;
3	3, 7, 13, 29, 30;
....	

After observing the completed table, prove that,

046 cont'd:

(ii) $p - 1$ is divisible by n ;

(iii) $10^{p-1}-1$ is divisible by p. (This is a particular case of Fermat's theorem.) (Remainder table need not be sent in).

ANSWER: (i) Let the remainders in the division process for $\frac{1}{p}$ be, in the order in which they occur, $1 = r_1, r_2, r_3, .. r_n$, $1 = r_{n+1}, \dots$. Thus $10.r_k = p.q_k + r_{k+1}$, $(k = 1,2, \dots , n)$ where q_k is the digit in the kth place after the decimal point. Similarly, let the division process for $\frac{a}{p}$ produce remainders $a = R_1, R_2, \dots , R_k, \dots$ in order, so that $10R_k = pQ_k + R_{k+1}$, $k = 1,2, \dots$. Then R_k is also the remainder when $a.r_k$ is divided by p .

[Proof - By induction. Obvious when $k=1$ since $a\,r_1 = a = R_1$. The identity $a\,r_{k+1} - R_{k+1} = 10(a\,r_k - R_k) - pq_k + pQ_k$ shows that if $a\,r_k - R_k$ is divisible by p, so is $a\,r_{k+1} - R_{k+1}$]

It is now clear that $R_h = R_k$ if and only if $a\,r_h$ and $a\,r_k$ leave the same remainder on division by p, i.e., if and only if p divides $a(r_h - r_k)$. Since p is prime, and a and $r_h - r_k$ are each less than p, this is true only for $r_h = r_k$. Hence the period of the expansion of $\frac{a}{p}$ is exactly the same as that of $\frac{1}{p}$, for every positive integer a less than p.

Furthermore, the remainders which occur for $\frac{a}{p}$ also occur for $\frac{R_2}{p}$, $\frac{R_3}{p}$, and $\frac{R_n}{p}$, n times in all. Since given a remainder R_k both R_{k-1} and R_{k+1} are determined uniquely

046 (i) Answer cont'd:

by $10\,R_{k-1} = pQ_{k-1} + R_k$ and $10\,R_k = pQ_k + R_{k+1}$, none of these remainders can occur in any fraction $\frac{b}{p}$ other than the n fraction already quoted.

(ii) As shown in (i), the p-1 distinct remainders 1,2,...,p-1 occur in disjoint sets of n . ("Disjoint" means that no element (i.e. remainder) occurs in two different sets). Hence p-1 is divisible by n .

iii) This should have read $10^{p-1}-1$ is divisible by p .

Since $\frac{a}{p} = .\overline{R_1\,R_2\,\ldots\,R_n} = \frac{A}{10^n-1}$ (see Vol.2,No.3, pp 4-5)

it is clear that p divides 10^n-1. However, n is a factor of p-1, say $p-1 = h.n$ and therefore

$10^{p-1}-1 = 10^{hn} - 1 = (10^n-1)\,(10^{(h-1)n}+10^{(h-2)n} + \ldots + 10^n+1)$,

which proves that 10^n-1 is a factor of $10^{p-1}-1$. Hence p divides $10^{p-1}-1$.

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047 Prove that the Fibonacci sequence,

$u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $u_5 = 5$, $u_6 = 8$, $u_7 = 13$ .... , where $u_{n+1} = u_n + u_{n-1}$ for $n \geq 2$,

has the following properties.

(i) $u_n^2 - u_{n-1}\,u_{n+1} = (-1)^{n-1}$

(ii) u_n is even if and only if n is a multiple of 3.

(iii) $u_n = u_{r+1}\,u_{n-r} + u_r\,u_{n-r-1}$, $1 \leq r \leq n-2$

(iv) u_k is a factor of u_n if and only if k is a factor of n .

047 cont'd:

ANSWER: All parts may be proved by the method of mathematical induction.

(i) For $n = 2$ this becomes $1^2 - 1.2 = (-1)^1$ which is obviously true.

Assume that $u_k^2 - u_{k-1} u_{k+1} = (-1)^{k-1}$ and consider

$$\begin{aligned} u_{k+1}^2 - u_k u_{k+2} &= u_{k+1}^2 - u_k (u_{k+1} + u_k) \\ &= u_{k+1} (u_{k+1} - u_k) - u_k^2 \\ &= u_{k+1} u_{k-1} - u_k^2 \\ &= -(-1)^{k-1} = (-1)^{(k+1)-1} \end{aligned}$$

Thus if the formula is true for $n = k$ it is also true for $n = k+1$, and this completes the proof.

(ii) It is to be proved that u_{3n} is even, and u_{3n-1} and u_{3n-2} are odd, for all positive integral n. This is true by inspection when $n = 1$. Assuming that u_{3k} is even and u_{3k-1} is odd it follows that $u_{3(k+1)-2} = u_{3k} + u_{3k-1}$ is odd, that $u_{3(k+1)-1} = u_{3k+1} + u_{3k}$ is odd, and that

$u_{3k} = u_{3k+1} + u_{3k+2}$ is even, which completes the proof.

(iii) When $r = 1$ this becomes $u_n = 1 \cdot u_{n-1} + 1 \cdot u_{n-2}$ which is true. Assume that $u_n = u_{k+1} u_{n-k} + u_k u_{n-k-1}$, $(k \leq n-3)$ and consider $u_{k+2} u_{n-k-1} + u_{k+1} u_{n-k-2}$.

047 (iii) Answer cont'd:

This equals $(u_{k+1} + u_k)\, u_{n-k-1} + u_{k+1}\, u_{n-k-2}$

$$= u_{k+1}\,(u_{n-k-1} + u_{n-k-2}) + u_k\, u_{n-k-1}$$

$$= u_{k+1}\, u_{n-k} + u_k\, u_{n-k-1}$$

$$= u_n \;.$$

Hence the result.

(iv) This is not true if $k = 2$ since $u_2 = 1$, a factor of every integer. For other values of k the statement is true and may be proved as follows.

Using (iii) $u_n = u_{k+1}\, u_{n-k} + u_k\, u_{n-k-1}$. Since u_k and u_{k+1} are relatively prime (Proof: Any prime factor common to u_{k+1} and u_k is also a factor of $u_{k-1} = u_{k+1} - u_k$, and similarly of u_{k-2}, u_{k-3}, etc. Eventually it is seen to be a factor of $u_1 = 1$.), it follows that u_k divides u_n if, and only if, it divides u_{n-k}. Repeating this we see that u_k divides u_n if and only if it divides u_{n-2k}, u_{n-3k} etc. If $n = q.k + r$ where $0 < r < k$ we have shown that u_k divides u_n if and only if u_k divides $u_{n-qk} = u_r$. Since $o < u_r < u_k$, u_k cannot divide u_r and therefore it is not a factor of u_n. This leaves the case $n = q\,k$. Here we have shown that u_k divides u_n if and only if it divides $u_{n-(q-1)k} = u_k$. Since this is certainly so, we have u_k a factor of u_n if and only if k is a factor of n.

047 cont'd:

Correct: W.H. Wilson, P. Underwood. The latter noted the exceptional value of k, and gave a very clear proof of the difficult part (iv).

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048 For the Fibonacci sequence, if p is any prime number, there is a value of $n \leq p+1$ such that p is a factor of u_n.

ANSWER: Consider the sequence of remainders left when the terms of the Fibonacci sequence are divided successively by p. This sequence

$$r_1(=1),\ r_2(=1),\ r_3,\ r_4,\ \ldots,\ r_n,\ \ldots$$

has the property

$$r_{n+1} = r_n + r_{n-1} \quad \text{or} \quad r_{n+1} = r_n + r_{n-1} - p .$$

(If $p = 5$, the sequence of remainders is 1,1,2,3,0,3,3,1,4, 0,4,4,3, ...). We have to show that for each p, a nought occurs in this sequence.

First note that the sequence is recurring; for any two consecutive remainders determine uniquely the following remainder (and also the preceding one). Since there are only p different remainders $(0,1,2, \ldots p_{\cdot\cdot})$ there are only p^2 different possible pairs. Of the p^2+1 pairs r_1r_2, r_2r_3, ... $r_{p^2+1}\, r_{p^2+2}$ there must be at least one pair which is identical with an earlier one and the sequence recurs. Since a pair of remainders determines the preceding remainder it is easy to see that the periodic nature of the sequence of remainders extends right back to the beginning of the sequence. (By analogy with infinite decimals one might call the sequence a "pure" recurring sequence, not a "mixed" one). Suppose the period is n ($\leq p^2$ as shown above). Then $r_{n+1} = r_1 = 1$, and $r_{n+2} = r_2 = 1$. Hence $r_n = r_{n+2} - r_{n+1} = 0$, and we have shown that one of the first p^2 Fibonacci numbers is divisible by p . In fact, since the pair of remainders 0,0 , can never occur, we can immediately modify the above argument to improve the result to "one of the first $p^2 - 1$" Fibonacci numbers. However we are asked to do much more than this.

Let the first zero occur as r_m , and let $r_{m+1} = a = a\,r_1$. Then $r_{m+2} = 0 + a = a = a\,r_2$, $r_{m+3} = a\,r_1 + a\,r_2(-p) = a\,r_3$ or $a\,r_3 - p$, and in fact it is clear that r_{m+k} is equal to the remainder when $a\,r_k$ is divided by p . In particular r_{2m} is equal to $a.0$, so that every m^{th} Fibonacci number is divisible by p .

To prove that m is not greater than $p + 1$, consider the sequences obtained by multiplying $r_1, r_2, \ldots, r_m = 0, r_{m+1}$ in succession by $1,2,3, \ldots, p+1$. Thus

$$r_1, r_2, \ldots, r_m = 0, r_{m+1}$$

$$2r_1, 2r_2, \ldots, 2r_m = 0, 2r_{m+1}$$

$$3r_1, 3r_2, \ldots, 3r_m = 0, 3r_{m+1}$$

....

$$(p-1)r_1, (p-1)r_2, \ldots, (p-1)r_m = 0, (p-1)r_{m+1}$$

Replace each number by its remainder on division by p. It is clear that the only zeros that occur are the multiples of r_m . It is then not difficult to see that no consecutive pair in any row is identical with a different consecutive pair in the same,

048 Answer cont'd:

or in any other row (certainly not vertically below since $a\,r$ and $b\,r$ do not leave the same remainders on division by p if $a < b < p$; and remembering that a consecutive pair determines the succeding sequence of numbers uniquely, identical pairs occurring in different positions in two rows would mean that the column of multiples of r_m would not all be zero.)

In each row of this array there are m different consecutive pairs {e.g. $r_1 r_2, r_2 r_3, \ldots, r_m r_{m+1}$}. Since there are $p-1$ rows there are $(p-1)\,m$ different consecutive pairs altogether, and as we noted earlier, the maximum possible number of consecutive pairs which can possibly occur is p^2-1 .

$$\text{i.e. } (p-1)\,m \le p^2-1 \quad \text{and} \quad m \le p+1 .$$

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049 Construct a Hadamard configuration for $m = 3$, $n = 11$.

ANSWER: Given 11 objects (e.g. 0,1,2, ... , 9,10) we have to find 11 subsets each containing 5 objects, and such that any two subsets have exactly 2 objects in common. It is not difficult to do this by trial and error, and there are a very large number of possible solutions. One interesting method (which works whenever $4m-1$ is a prime, although we shall make no attempt to prove this) is the following:- For the first subset take the remainders when the first five squares are divided by 11.

$S_1 = \{1, 3, 4, 5, 9\}$. For S_2 add one to each of these numbers, thus;

$S_2 = \{2, 4, 5, 6, 10\}$. Do this again to obtain S_3 , replacing the number 11, which is not one of the set of objects, by 0 . $S_3 = \{3, 5, 6, 7, 0\}$. If this procedure is repeated until 11 sets are obtained, it will be found that a Hadamard configuration results.

Correct: W.H. Wilson, P. Underwood, S. Byrnes.

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050 SOLUTION TO CROSS NUMBER PUZZLE

0	5	4	5	0	1
2	1	1			1
1	0		1		0
3		1	2	1	0
1			5		2
2	1	1	4	0	

Across

1. $\frac{5}{31} = (.\overline{054501})_6$
5. $\frac{11}{13} = (.\overline{211})_3$
6. $\frac{2}{3} = (.\overline{10})_2$
8. $\frac{3}{5} = (.\overline{1210})_3$
9. $\frac{5}{11} = (.\overline{21140})_5$

Down

1. $\frac{2}{13} = (.\overline{021312})_4$
2. $\frac{14}{19} = (.\overline{510})_7$
3. $\frac{5}{7} = (.\overline{41})_6$
4. $\frac{2}{11} = (.\overline{11002})_3$
7. $\frac{1}{5} = (.\overline{1254})_7$

Correct: D. Hermann, G. Matheson, P. Underwood and S. Byrnes.

ANSWERS

Two of a Kind (p.14) 1) 86 matches, since in each match one entrant is eliminated and eventually all save the winner have been eliminated.

2) 27 operations are always required, since each break increases the number of pieces by one.

Cross Number Puzzle (p.13)

Across: $2 + 5 - 4 = 3$; $6 \times 1 - 2 = 4$; $4 \times 5 \div 5 = 4$.

Down: $2 \times 6 \div 4 = 3$; $5 + 1 - 5 = 1$; $4 \times 2 - 5 = 3$.