SOLUTIONS OF JUNIOR MATHEMATICS COMPETITION QUESTIONS.

Question 1. The front tyres of a car last for 25,000 miles, the back tyres for 15,000 miles. After a number of miles the back and front tyres are in terchanged. What is the maximum mileage obtainable without buying new tyres?

You may assume that the wear on the tyres is proportional to the distance travelled.

ANSWER: Call 5,000 miles one unit of distance. After travelling x units, the fraction of allowable wear on the front tyres (A) is $\frac{x}{5}$, and on the rear tyres (B) is $\frac{x}{3}$.

After interchanging the tyres and travelling a further y units these tyres wear a further fraction of their "life" equal to $\frac{y}{3}$ and $\frac{y}{5}$ respectively.

Thus the tyres A have now expended

$$\left(\frac{x}{5} + \frac{y}{3} \right) \text{ of their life}$$

and the tyres B

$$\left(\frac{x}{3} + \frac{y}{5} \right) \text{ of their life .}$$

We have to find the maximum value of $(x+y)$

ANSWER to Question 1. cont'd

such that $\frac{x}{5} + \frac{y}{3} \leq 1$

and $\frac{x}{3} + \frac{y}{5} \leq 1$

But $x + y = \frac{15}{8}\left(\frac{x}{5} + \frac{y}{3}\right) + \frac{15}{8}\left(\frac{x}{3} + \frac{y}{5}\right) \leq \frac{15}{8} + \frac{15}{8} = \frac{15}{4}$

The upper limit is attained by taking $x = y = \frac{15}{8}$.

Thus the maximum mileage attainable is $\frac{15}{4} \times 5000$ miles = 18,750 miles, and this is attainable only by changing tyres at the half way mark, after 9,375 miles.

Find the Ladies. (p. 14)

Solution. If a man buys x articles at x/- each and his wife y articles at y/- each, he spends $(x^2 - y^2)$ shillings more than she. For each of the three pairs

$$x^2 - y^2 = 63. \text{ Hence } (x-y)(x+y) = 63.$$

Both factors are whole numbers, the first smaller than the second, so that there are only three possibilities corresponding to the factorizations of 63 into 1.63 or 3.21 or 7.9. If $x-y = 1$ and $x+y = 63$ we easily obtain $x = 32$, $y = 31$. The other possibilities give $x = 12$, $y = 9$ or $x = 8$, $y = 1$. It is now easy to see that Dorothy bought 9 and Charles 8, and since Fanny bought 1 article she is Charles's wife. Dorothy's husband bought 12 articles, so is not Arthur; he must be Benjamin. This leaves Arthur and Edith as the remaining pair.

QUESTION 2. (a) For what integers n is $(n^2-1)(n^2+3)$ divisible by 64?

(b) Prove that if 5 distinct positive integers are all ≤ 30, there exist two, x and y, such that $x < y \leq 2x$.

ANSWER:(a) If n is even, both factors are odd, and the product is not divisible by 64. Hence n is odd.

Put $n = 2k + 1$ where k is an integer. Then $(n^2-1) = 4k(k+1)$ and $n^2 + 3 = 4\,[k(k+1) + 1]$

$$\text{Hence } (n^2-1)(n^2+3) = 16\,k\,(k+1)\,[k(k+1)+1]$$

Since one of k and $k + 1$ is even, the quantity is square brackets is odd. The L.H.S. is divisible by 64 if and only if $k(k+1)$ is divisible by 4 i.e. if $k = 4k'$, or $4k'-1$, where k' is an integer. Since $n = 2k+1$ we now have

$$n = 8k' + 1 \text{ or } 8k' - 1.$$

The required integers are those which differ from a multiple of 8 by 1.

ANSWER:(b) Let x_1, x_2, x_3, x_4, and x_5 be the integers in increasing order of magnitude. We shall produce a contradiction by assuming that the statement is false. It is false only if

$$x_{i+1} > 2x_i \quad \text{for } i = 1, 2, 3 \text{ and } 4.$$

i.e. $x_2 > 2x_1 \geq 2.1$ Hence $x_2 \geq 3$.

$x_3 > 2x_2 \geq 2.3$ Hence $x_3 \geq 7$.

$x_4 > 2x_3 \geq 2.7$ Hence $x_4 \geq 15$.

$x_5 > 2.15 \pm 30$. But we are given $x_5 \leq 30$.

This contradiction establishes the result.

QUESTION 3. A country introduces a new currency and issues only two different kinds of coins: 5 cent pieces and 8 cent pieces. Show that any amount (in cents) can be paid for exactly, by receiving change if necessary. For instance, 4 cents can be paid by tendering three 8 cent pieces and receiving four 5 cent pieces in change. What is the largest amount which cannot be paid for exactly without receiving change?

ANSWER: One cent can be paid by tendering 2 eight cent pieces and receiving 3 five cent pieces in change. Hence x cents can be paid by tendering $2x$ eight cent pieces and receiving $3x$ five cent pieces in change. This completes the first part of the problem.

Let $x = 5q + r$ where $0 \leq r < 5$.

If $r = 0$, x cents can be paid by tendering q five cent pieces. If $r = 1$, x cents can be paid by tendering 2 eight cent pieces and $q - 3$ five cent pieces, provided $x \geq 16$. i.e. provided $x > 11$.

ANSWER 3. cont'd

If $r = 2$, x cents can be paid by tendering 4 eight cent pieces and $(q-6)$ five cent pieces, provided $x \geq 32$, $x > 27$.

If $r = 3$, x cents can be paid by tendering 1 eight cent piece and $(q - 1)$ five cent pieces provided $x \geq 8$.

If $r = 4$, x cents can be paid by tendering 3 eight cent pieces and $(q-4)$ five cent pieces provided $x \geq 24$.

Hence for every $x > 27$, it is possible to pay x cents without receiving change. Since 27 cents cannot be tendered exactly, this is the largest such sum.

QUESTION 4. How many different (non-congruent) triangles are there with sides of integer lengths and perimeter 24? How many with perimeter 60?

ANSWER: Let the lengths of the sides of the triangle be a, b and c where $a \leq b \leq c$ and all are integers. If P is the perimeter of the triangle, then $c < \frac{P}{2}$ (since the shortest two sides have a combined length in excess of the third side). Clearly also $c \geq \frac{P}{3}$. Thus c, the length of the longest side, is an integer in the range $\frac{P}{3} \leq c < \frac{P}{2}$. For each value of c in this range,

$a + b = P - c$ and since $b \geq a$ it follows that $b \geq \frac{P-c}{2}$. Thus the second longest side is an integer in the range $\frac{P-c}{2} \leq b \leq c$.

Having chosen both c and b the smallest side a is uniquely determined.

ANSWER to Question 4. con'd

When $P = 60$ the allowable values of c are the integers from 20 to 29 (since $\frac{60}{3} \leq c < \frac{60}{2}$).

For each of these in turn we calculate the number of allowable values of b; e.g. when $c = 25$ we have

$$\frac{60 - 25}{2} \leq b \leq 25 ;$$ and there are 8 possible values of b. We find that for $c = 20, 21, \ldots, 29$, the numbers of allowable values of b are 1, 2, 4, 5, 7, 8, 10, 11, 13, and 14 respectively. The sum of these, 75, is the total number of different triangles obtainable.

If $P = 24$ a similar but shorter calculation shows that 12 triangles satisfy the conditions.

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Hadamard Configuration

Readers of Parabola Vol. 2 No.3 may remember Professor Szekeres's article on the Hadamard Problem, and his observation that no Hadamard configuration with $m = 29$ had yet been found. A construction of such a configuration has just been announced in the Bulletin of the American Mathematical Society, by an American mathematician, L.D. Baumert. The smallest value of m for which no configuration is known is now $m = 47$.

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QUESTION 5.

(i) Let a,b,c,d be positive numbers, $a < b < c < d$. Arrange the fractions

$$\frac{c}{d}, \quad \frac{b+c}{a+d}, \quad \frac{b+c}{d-a}, \quad \frac{c-b}{a+d}, \quad \frac{c-b}{d-a}$$

in increasing order of magnitude and prove that the order is independent of the actual values of a,b,c,d.

(ii) Is it possible to place $\frac{b}{a}$ in a similar fashion among the previous fractions?

ANSWER: (i) The order is

$$\frac{c-b}{d+a} < \frac{c-b}{d-a} < \frac{c}{d} < \frac{b+c}{a+d} < \frac{b+c}{d-a}$$

[This is most quickly obtained by giving special values to the symbols e.g. $a=1$, $b=2$, $c=3$, $d=4$. However, it will still be necessary to prove that the order does not depend on the special values chosen, as follows.]

The first inequality is obvious since the two numerators are equal, but the second denominator is smaller. A similar observation applies to the last inequality.

To prove the second inequality, note that

$$\frac{c-b}{d-a} < \frac{c}{d} \leftrightarrow d(c-b) < c(d-a) \leftrightarrow -bd < -ac$$

$$\leftrightarrow bd > ac$$

The last statement is obviously true since the numbers are all positive, $b > a$, and $d > c$.

A similar argument establishes the third inequality.

ANSWER to Question 5 cont'd

(ii) It is equally easy to show that

$$\frac{b+c}{a+d} < \frac{b}{a}$$

so that it only remains to test whether $\frac{b}{a}$ is always on the same side of $\frac{b+c}{d-a}$.

If a,b,c,d are respectively 1,2,3 and 4 then

$$\frac{b}{a} = 2 > \frac{5}{3} = \frac{b+c}{d-a} ;$$

whilst if a,b,c,d are respectively 2,3,4 and 5 then

$$\frac{b}{a} = \frac{3}{2} < \frac{7}{3} = \frac{b+c}{d-a} .$$

The value of $\frac{b}{a}$ may either exceed or fall short of $\frac{b+c}{d-a}$, depending on the values given to the symbols, so that it is not possible to uniquely order the fractions if

$$\frac{b}{a}$$

is included.

QUESTION 6. Given four circles such that every three of them contain at least one common interior point, show that there is a point which is in the interior of all four circles.
An interior point is one which is inside the circle but not on the circumference.

ANSWER: Note that if A and B are both interior points of a circle, so are all points on the line segment AB. As a consequence,

(i) if A and B are both interior points of two circles, so are all points on the line segment AB. It is also obvious that

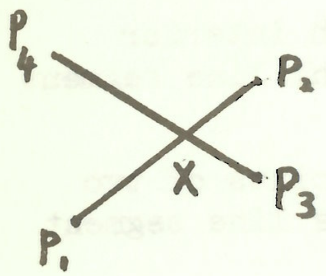
(ii) if A,B and C are all interior points of a circle, then all points inside the triangle ABC are interior points of the circle.

Label the four circles C_1, C_2, C_3 and C_4 and find points P_1, P_2, P_3 and P_4 in accordance with the data, such that P_1 is an interior point of C_2, C_3 and C_4; in fact, P_k is an interior point of every circle except C_k.

Now if P_4 is an interior point of the $\Delta P_1 P_2 P_3$ it is actually in all four circles, and we are finished. [By its definition it is in C_1, C_2, and C_3; and, since P_1, P_2 and P_3 are all interior points of C_4, so is P_4, by (ii)]

ANSWER to Question 6 con'd.

Suppose, then, that none of the points P_k is an interior point of the triangle formed by the other three. Then P_1, P_2, P_3, P_4 are vertices of a convex quadrilateral. Consider X, the point of intersection of the diagonals of this quadrilateral. Two applications of (i) now show that X is an interior point of all four circles. [For example,



let one diagonal be $P_1 P_2$ and the other $P_3 P_4$. Then both P_1 and P_2 lie in both the circles C_3 and C_4 and, by (1), so do all points on $P_1 P_2$ including X. Similarly, since X is on $P_3 P_4$ it is an interior point of both the circles C_1 and C_2.]

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ANSWERS TO PUZZLES.

Lost Division (p. 10)

Answer. 1,000,989 ÷ 99 = 10,111

Painted Faces (p. 6)

Answers for 3" cube. 8, 12, 6, and 1 respectively.

Answers for n" cube. 8, $12(n-2)$, $6(n-2)^2$ and $(n-2)^3$ respectively.

Multiplication Puzzle. (p. 19) A = 2, B = 6, C = 5, D = 1, E = 7.