Parabola Volume 3, Issue 2 (1966)

SOLUTIONS OF PROBLEMS IN PARABOLA, VOL.3, No.1.

J51 Prove that a number which consists of 3^n equal digits is divisible by 3^n. [e.g. if n=1, 555 is divisible by 3 ; if n=2, 888, 888, 888 is divisible by 9, and so on].

ANSWER: It is clearly sufficient to prove the result when the digit is 1. Use mathematical induction. The result is true when $n = 1$ since 111 is divisible by 3^1. Assume the result is true for $n = k$ i.e. that $x = 111.......1$ (3^k digits) is divisible by 3^k.

Consider $y = 111......1$ (3^{k+1} digits)

$$y = x + x.10^{3^k} + x(10^{3^k})^2$$

$$= x\,[1 + u + u^2] \text{ where } u = 10^{3^k}$$

$$= x\,[(u-1)^2 + 3u]$$

Since $u - 1 = 999......9$, 3 is a factor of both terms in the bracket, and by our assumption 3^k is a factor of x. Hence 3^{k+1} is a factor of y. Hence the result is true when $n = k+1$ if it is true when $n = k$. Since it is true for $n = 1$ it is true for all n.

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J52. Prove that if the number x is given by 0.999 .., where there are at least 50 nines, then $\sqrt{x}$ also has 50 nines at the beginning.

ANSWER: For any positive number $x<1$, $\sqrt{x}$ is also positive and <1.

Hence $\sqrt{x} \cdot \sqrt{x} < \sqrt{x} \cdot 1$ i.e. $x < \sqrt{x} < 1$.

Thus the square root of a number less than 1 lies between that number and 1, and the stated result follows immediately.

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J53. If $x + y + z = 0$ simplify the following expression:

$$\left(\frac{y - z}{x} + \frac{z - x}{y} + \frac{x - y}{z}\right) \left(\frac{x}{y - z} + \frac{y}{z - x} + \frac{z}{x - y}\right)$$

ANSWER: Put $X = y-z$; $Y = z-x$; $Z = x-y$ and note that $X + Y + Z = 0$.

Then the first factor is

$$\left(\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z}\right) = \left(\frac{X + Y + Z}{x} + Y\left(\frac{1}{y} - \frac{1}{x}\right) + Z\left(\frac{1}{z} - \frac{1}{x}\right)\right)$$

$$= 0 + \frac{YZ}{xy} - \frac{ZY}{xz}$$

$$= \frac{YZ}{x}\left(\frac{1}{y} - \frac{1}{z}\right)$$

$$= -\frac{XYZ}{xyz}$$

J53 Answer cont'd:

Similarly the second factor is $\left(\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z}\right)$

$$= \left(\frac{x+y+z}{X} + y\left(\frac{1}{Y} - \frac{1}{X}\right) + z\left(\frac{1}{Z} - \frac{1}{X}\right)\right)$$

$$= \left(0 + \frac{-3yz}{YX} + \frac{3yz}{ZX} \right)$$

(since $X - Y = x + y - 2z = x + y + z - 3z = -3z$) etc.

$$= \frac{3yz}{X}\left(\frac{1}{Z} - \frac{1}{Y}\right) = \frac{3yz}{X}\;\frac{-3x}{YZ} = \frac{-9xyz}{XYZ}$$

The product is thus $\frac{-XYZ}{xyz} \times \frac{-9xyz}{XYZ} = 9$

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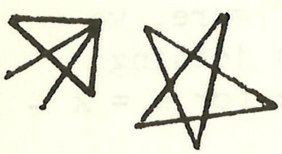
054

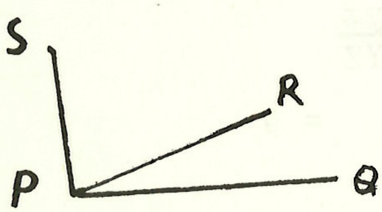


These diagrams show two ways of drawing 5 line segments connecting the vertices of a convex pentagon with the property that every pair of lines intersect. It is impossible to draw 6 line segments (each an unproduced side or diagonal of the pentagon) with this property. Prove that in fact it is not possible to draw (n + 1) sides or diagonals of a convex polygon with n vertices in such a way that every pair intersect.

ANSWER: The result is obvious if n = 3. Suppose we show that if the result is false when n = k, it is false for n = k-1. It will follow that it cannot be false for any value of n(>3) since this would imply falsity for n = 3.

054 Answer cont'd:

Suppose it is not true for n = k i.e. that (k+1) line segments with the desired property can be drawn connecting the k vertices of a convex polygon. In such a figure it is possible to find a vertex which is the end point of at least 3 line segments, (since there are 2×(k+1) ends of line segments distributed amongst k vertices, an average of more than 2 per vertex). Let P be such a vertex and PQ, PR, and PS line segments terminating at P. Because of the convexity of the polygon, $S\hat{P}Q$ is less than 180° and it is clearly impossible to draw a line segment terminating at R which intersects both PQ and PS. Thus RP is the only line segment which terminates at R. If now the segment RP and the vertex R are omitted from the figure, we are left with k intersecting line segments joining k-1 vertices. Hence the result is not true for n = k - 1.



Good attempts at this difficult problem were sent by D. Shillito (Trinity Grammar School); J. Tang (Waverley College); P. Jablon (Telopea Park H.S.); and S. Byrnes (Macquarie B.H.S.).

Lost Division

Can you replace the crosses by digits to make the division correct?

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++ )+++++++ (+++1+
     ++
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     +++
      ++
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      +++
       ++
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        ++
        ++
       ----

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(Answer p.32)

055 Show that the binomial coefficients, ${}^nC_r$, defined by $(a+b)^n = a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots\ldots$

$+ {}^nC_r\, a^{n-r}\, b^r + \ldots\ldots + {}^nC_n\, b^n$,

satisfy the following identities for any positive integer n.

(i) $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$.

(ii) $\quad {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots \qquad = 0$.

(iii) $\quad ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n\, {}^{2n}C_n$.

(iv) $\quad ({}^{2n+1}C_0)^2 - ({}^{2n+1}C_1)^2 + ({}^{2n+1}C_2)^2 \ldots - ({}^{2n+1}C_{2n+1})^2 = 0$.

(v) $\quad ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$.

ANSWER: (i) Put $a = b = 1$

(ii) Put $a = 1 \quad b = -1$

(iii) Consider $(1-x^2)^{2n} = (1-x)^{2n}\,(x+1)^{2n}$

$$ {}^{2n}C_0 - {}^{2n}C_1\, x^2 + \ldots\ldots + (-1)^n\, {}^{2n}C_n x^{2n} + \ldots + {}^{2n}C_{2n} x^{4n} $$

$$ = ({}^{2n}C_0 - {}^{2n}C_1\, x + {}^{2n}C_2 x^2 - \ldots + {}^{2n}C_{2n} x^{2n}) $$

$$ ({}^{2n}C_0 x^{2n} + {}^{2n}C_1 x^{2n-1} + {}^{2n}C_2\, x^{2n-2} \ldots\ldots\ldots + {}^{2n}C_{2n}\, 1) $$

055 Answer cont'd

Now two polynomials are equal for every value of x only if they are identical, so that we can equate the coefficients of any power of x in the two sides of this equation. The coefficients of x^{2n} are

$$(-1)^n \, {}^{2n}C_n \quad \text{and} \quad ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 \ldots + ({}^{2n}C_{2n})^2$$

in the L.H.S. and the R.H.S. respectively.

(iv) Either as for (iii) compare the coefficient of x^{2n+1} in both sides of $(1-x^2)^{2n+1} = (1-x)^{2n+1}(x+1)^{2n+1}$

or, alternatively, observe that ${}^nC_r = {}^nC_{n-r}$ (by equating the coefficients of x^r in $(1+x)^n$ and $(x+1)^n$). It follows that the terms in (iv) cancel out in pairs.

(v) Consider $(1+x)^{2n} = (1+x)^n(x+1)^n$

$$\sum_{r=1}^{2n} {}^{2n}C_r \, x^r = \left(\sum_{i=1}^{n} {}^nC_i x^i\right) \left(\sum_{j=1}^{n} C_j \, x^{n-j}\right)$$

Equating the coefficients of x^n gives the result.

Correct: P. Jablon; C.Cosgrove; J. Lang.

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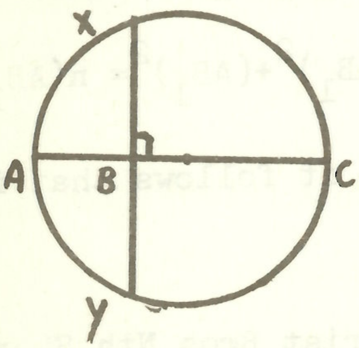
056 Duplicate, and generally n-plicate the square. That is, given a square in the plane, construct one with an area n times the original square.

ANSWER:- Let AB be a line segment whose length is equal to the side of the given square. Produce AB to C making BC twice (or n times) the length of AB. Let the perpendicular to AC at B intersect the circle whose diameter is AC at X and Y. Then the square whose side is BX has an area twice (or n times) that of the square on AB.

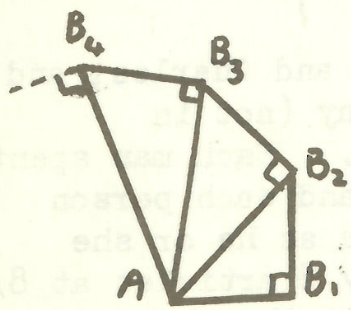


Proof: XB.BY = AB.BC. (1) [e.g. from $\frac{BY}{BC} = \frac{BA}{BX}$ in similar triangles BAX and BYC]

But XB = BY since the perpendicular from the centre bisects the chord XY. Also BC = 2.AB (or n.AB). Substitution in (1) now gives

$$BX^2 = 2AB^2 \text{ (or } n.AB^2\text{)}$$

Alternative solution.



Let AB_1 be a line segment whose length is equal to the side of the given square, which we shall call 1 unit of length. Construct the accompanying figure, making each of the segments $B_1 B_2$, $B_2 B_3$, ..., $B_{n-1} B_n$, of unit length.

The square whose side is AB_n has an area equal to n times that of the square on AB_1.

056 Answer cont'd

Proof: If $(AB_{n-1})^2 = (n-1(AB_1)^2$, Pythagoras'theorem applied to the right angled triangle $AB_{n-1}B_n$ gives

$$(AB_n)^2 = (AB_{n-1})^2 + (B_{n-1}B_n)^2 = (n-1)(AB_1)^2 + (AB_1)^2 = n(AB_1)^2$$

Since the statement is true for n = 2, it follows that it is true for all positive integers n.

Correct:- P. Jablon, J.Long, W.Wu (Marist Bros Nth Shore H.S.) O. Nash (Parkes H.S.), B. Patterson (Asquith B.H.S.), S. Byrnes, P.W. Pearson (De La Salle College, Ashfield); C. Cosgrove (who also indicated how the construction could be performed if n was not an integer, but a positive fraction).

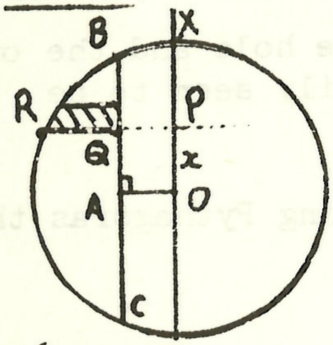
Find the Ladies. (Solution p. 24)

Three men, Arthur, Benjamin and Charles, and their wives Dorothy, Edith and Fanny (not in corresponding order)went to market. Each man spent three guineas more than his wife, and each person spent as many shillings per article as he or she bought articles (e.g. one might buy 8 articles at 8/- each) Arthur bought one more article than his wife; Dorothy bought one more article than Charles; Fanny spent only one shilling.

Pair each man with his wife.

057 A cylindrical hole is drilled through the centre of a sphere. If the hole is 6" long find the volume of the remaining portion of the sphere.

ANSWER: The wording of the question suggests that a numerical answer is possible, i.e. that the answer is independent of the radius of the sphere. If this is so, it is easy to find the answer by considering the limiting case in which the sphere has diameter 6", the hole having zero width. The volume remaining is the total volume of the sphere viz.



36π cubic ins. In order to prove that the volume remaining is the same regardless of the radius of the sphere, a calculus method is suitable. Consider the figure, which shows a cross section through the centre of the sphere containing the axis OX of the cylindrical hole. The "remaining portion of the sphere" is obtained by revolving the segment CQBR about the axis OX.

Let PQR be a straight line perpendicular to OX at a distance of x ins above O, and let $V(x)$ be the volume obtained by revolving CQR about OX.

Now $\dfrac{dV}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta V}{\delta x}$ where δV is the volume of

the washer shaped region of thickness δx obtained by revolving the shaded region about OX.

057. Answer cont'd.

The area $A(x)$ of the lower face of the washer is

$$\pi(\sqrt{r^2 - x^2})^2 - (\sqrt{r^2 - 3^2})^2$$

[since the radii PQ and PR of the hole and the outer circumference of the washer are easily seen to be

$\sqrt{r^2 - 9}$ and $\sqrt{r^2 - x^2}$ by applying Pythagoras theorem to the triangles OAB and OPR].

Thus $A(x) = \pi(9-x^2)$.

Hence $\delta V \approx \pi(9-x^2)\ \delta x$ where it is obvious that the approximation becomes very good as δx approaches 0.

Hence $\dfrac{dV}{dx} = \pi(9-x^2)$

Integrating we obtain $V(x) = \pi(9x - \dfrac{x^3}{3}) + C$

where C is a constant of integration which may be evaluated by noting that $V(-3) = 0$. This gives $C = 18\pi$. The required volume is now $V(3) = 36\pi$.

Correct: C. Evers (Vaucluse B.H.S.), D. Nash,
M.B. Plater (Macquarie B.H.S.)
J.R. Hughes (Sydney Grammar)
S. Byrnes, C. Cosgrove, D. Shillito.

058 Question 4. Find two different braids σ and τ on 3 strings such that

$\sigma \times \sigma = \tau \times \tau \times \tau$

Can you find two such braids which generate B_3 that is, with the extra property that every braid in B_3 can be written as a product of the braids σ, τ and their inverses σ^{-1} and τ^{-1}. (The braids σ_1 and σ_2 defined in the article generate B_3, but do not satisfy the additional property.)

ANSWER: Let $\sigma = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ (see p. 8 last issue.) and let $\tau = \sigma_1 \sigma_2$. Then

$$\tau^3 = (\sigma_1 \sigma_2)(\sigma_1 \sigma_2)(\sigma_1 \sigma_2)$$
$$= (\sigma_1 \sigma_2 \sigma_1)(\sigma_2 \sigma_1 \sigma_2) = \sigma^2$$

Also $\tau^{-1} = (\sigma_2^{-1} \sigma_1^{-1}$ so that

$$\tau^{-1} \sigma = (\sigma_2^{-1} \sigma_1^{-1})(\sigma_1 \sigma_2 \sigma_1) = \sigma_1$$

and $\sigma\tau^{-1} = (\sigma_2 \sigma_1 \sigma_2)(\sigma_2^{-1} \sigma_1^{-1}) = \sigma_2$

Similarly $\sigma_1^{-1} = \sigma^{-1} \tau$

and $\sigma_2^{-1} = \tau\sigma^{-1}$. Since every braid in B3 can be

058 Answer (Qst. 4) cont'd

written as a product of σ_1, σ_2, σ_1^{-1} and σ_2^{-1} and each of these can be written as a product of σ, τ, σ^{-1} and τ^{-1} it is clear that σ and τ generate B_3.

Question 8. Show how the leather braid shown can be produced from the trivial braid using the operations T_1 and T_2 and their inverses.

ANSWER. In terms of the braids σ_1 and σ_2 the required leather braid is equal to $(\sigma_2^{-1}\ \sigma_1)^6$,

whilst $T_1 = \sigma_2^{-1}\ \sigma_2^{-1}\ T_2 = \sigma_1\ \sigma_1$,

$T_1^{-1} = \sigma_2\ \sigma_2$, $T_2^{-1} = \sigma_1^{-1}\ \sigma_1^{-1}$.

From $\sigma_1\ \sigma_2\ \sigma_1 = \sigma_2\ \sigma_1\ \sigma_2$, we obtain

$$\sigma_2\ \sigma_1 = \sigma_1^{-1}\ \sigma_2\ \sigma_1\ \sigma_2 \quad (1)$$

and from $\sigma_1^{-1}\ \sigma_2^{-1}\ \sigma_1^{-1} = \sigma_2^{-1}\ \sigma_1^{-1}\ \sigma_2^{-1}$

$$\sigma_2^{-1}\ \sigma_1^{-1} = \sigma_1^{-1}\ \sigma_2^{-1}\ \sigma_1^{-1}\ \sigma_2 \quad (2)$$

058 (Question 8) Answer cont'd

Consider $(\sigma_2^{-1}\sigma_1)^3 = (\sigma_2^{-1}\ {}_1)(\sigma_2^{-1}\sigma_1)(\sigma_2^{-1}\sigma_1)$

$${}_2^{-1}(\sigma_2^{-1}\sigma_2)\sigma_1\ \sigma_2^{-1}\sigma_1\ \sigma_2^{-1}(\sigma_1^{-1}\sigma_1)\sigma_1$$

$$= T_1(\sigma_2\ \sigma_1)(\sigma_2^{-1}\sigma_1)(\sigma_2^{-1}\sigma_1^{-1})T_2$$

$$= T_1(\sigma_1^{-1}\sigma_2\sigma_1\sigma_2)(\sigma_2^{-1}\sigma_1)(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2)T_2$$

$$= T_1\ \sigma_1^{-1}\ (\sigma_2\ \sigma_1)(\sigma_2^{-1}\ \sigma_1^{-1})\sigma_2\ \ T_2$$

$$= T_1\ \sigma_1^{-1}(\sigma_1^{-1}\sigma_2\ \sigma_1\ \sigma_2)\sigma_2^{-1}\ \sigma_1^{-1}\ \sigma_2\ T_2$$

$$= T_1\ T_2^{-1}\ \ T_1^{-1}\ T_2$$

Hence $(\sigma_2^{-1}\sigma_1)^6 = (T_1T_2^{-1}\ T_1^{-1}\ T_2)^2$

$$= T_1T_2^{-1}\ T_1^{-1}\ \ T_2T_1T_2^{-1}\ T_1^{-1}\ T_2$$

Correct: S. Byrnes.

Multiplication Puzzle

Replace the letters by digits so that the multiplication is correct.

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  AB
  AB
 ---
 DCB
 CA
 ---
 BEB
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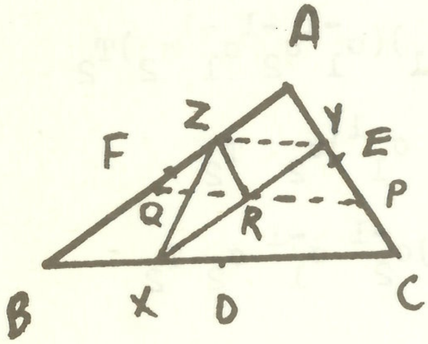
(Answer, p.32)

059. The triangle ABC is divided into four smaller triangles by means of the points X, Y, Z on the sides AB, BC, CD respectively. Prove that the middle traingle XYZ cannot have the smallest area (though it can be equal in area to the smallest corner triangle in certain cases).

ANSWER. This is difficult. Let DEF be the midpoints of BC, CA, and AB respectively. If X,Y,Z coincide with D,E,F then all four triangles have the same area. Otherwise we consider two cases.



Case I. Two of the vertices X,Y,Z lie on the perimeter of one of the triangles AFE, BDF and CED. Relabelling the figure if necessary we may suppose that Y lies in AE and Z in AF. (We do not rule out the special cases when Y coincides with E, or Z with F, or both).

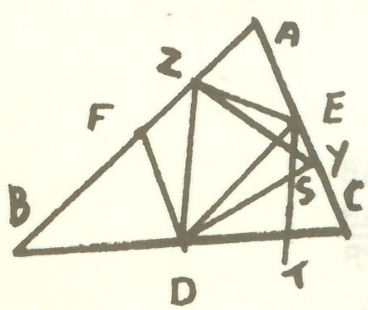
It is easy to show now that $\Delta AYZ < \Delta XYZ$. Double AY at P and AZ at Q, and note that P and Q lie in the segments AC and AB, and that PQ is parallel to YZ. Then if YX intersects PQ at R we have

$\Delta AYZ = \Delta PYZ$ (equal bases AY and PY and same vertex).

$= \Delta RYZ$ (on same base and between same parallels).

$\leq \Delta XYZ$ [Equality applies only if $Y \equiv E$ and $Z \equiv F$]

The only other possibility is



Case II. Each corner triangle AFE, BDEF, and CED has one X,Y,Z on its perimeter. We may label the figure so that Z lies in AF, Y in EC and X in BD. (See Diag.) Then

$\Delta DEF = \Delta DEZ < \Delta DYZ < \Delta XYZ$

059 Answer cont'd

(The equality follows since ΔDEF and ΔDEZ have the same base DE and lie between the same parallels. To show that $\Delta DEZ < \Delta DYZ$ draw $ET \| DZ$ and cutting DY at S. It is easy to see that

$$D\hat{E}T = B\hat{Z}D < B\hat{F}D = D\hat{E}C$$ so that S lies between D and Y.

Then $\Delta DEZ = \Delta DSZ < \Delta DYZ$. A similar argument establishes the second inequality).

Hence $\Delta XYZ > \frac{1}{4} \Delta ABC$ and it follows that it cannot be be the smallest of the four triangles AYZ, BZX, CXY and XYZ.

EDITORIAL

It has become an established custom to present in Parabola solutions of the problems set in the I.B.M. sponsored School Mathematics Competition. Answers for the Junior Division Examination appear in this issue (pp. 23-32); the Senior Division problems will be discussed in the next, together with the names of the winners of each division.

To make room for these solutions, the next of Professor Szekeres's articles on unsolved problems has been deferred to a later date.

The author of the article on the Pigeon Hole Principle Dr. J.D. Dixon, is a senior lecturer in pure mathematics, and an expert in the fields of modern algebra and the theory of numbers.

060 All entrants for a certain school chess tournament were from form IV or form V. There were nine times as many from the lower form, but between them they only scored 4 times as many points as the entrants from form V. The tournament was conducted on the Round Robin system (i.e., each entrant played all the others once, scoring 1 for a win, $\frac{1}{2}$ for a draw and 0 for a loss). What was the winner's score?

ANSWER: Suppose there were x entrants from form V. Then there were 10x entrants altogether, and the total number of games played was

$$\frac{10x(10x-1)}{2}.$$

The form V entrants scored $\frac{1}{5}$ of the total number of points, namely

$$\frac{1}{5} \times \frac{10x(10x-1)}{2} = x(10x-1),$$

an average of (10x-1) points each. But a player can score (10x-1) points only if he wins every game. It follows that $x = 1$ (since two form V players cannot both win the game they play against each other), and that he wins every game (and hence, obviously, the tournament) scoring 9 points.

Correct: S. Byrnes, P.W. Pearson, J. Lang, P. Jablon.