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GRAECO LATIN SQUARES AND THE DESIGN OF EXPERIMENTS.

In 1782 the Swiss mathematician, Leonard Euler, perhaps the greatest mathematician of the eighteenth century (the only other serious contender for the title is the Frenchman, J.L. Lagrange), published a paper bearing the title "On a New Type of Magic Square". In this paper he made a conjecture about "Graeco-Latin squares" which remained a challenge to mathematicians for over 175 years; it was not until 1958 that three mathematicians in America, Bose, Shrikhande, and Parker, were able to show that Euler's guess was substantially incorrect.

A Latin square of order n, is an arrangement of n symbols in an n x n square, with the property that each symbol occurs exactly once in each row, and once in each column. Fig. 1 shows two examples of Latin squares of order 3, one in which the 3 symbols are the Latin letters a, b, and c, ; and the other using the first three Greek letters.

a	b	c
b	c	a
c	a	b

α	β	γ
γ	α	β
β	γ	α

Fig. 1 A.

These two Latin squares have the additional special property that, when they are superimposed each of the 3^2 possible ways of pairing a Latin letter with a Greek letter occurs exactly once. Because of this property they are called "orthogonal" Latin squares, and the combined array obtained by superimposing a pair of orthogonal Latin squares is termed a Graeco-Latin square.

aα	bβ	cγ
bγ	cα	aβ
cβ	aγ	bα

Fig. 2 A.

It is obvious that if a, b, and c, in Fig. 1 are replaced by say x, y, z, respectively, or by Δ, O, □, respectively, the result will still be a Latin square using the new symbols. We chose to use letters in our examples because it is then clear how the name Graeco-Latin square arose. However, it is more convenient and usual to use the first n natural numbers. In this notation it is merely repeating our previous example to say that Fig. 2 B. is a Graeco-Latin square resulting from superimposing the orthogonal Latin squares in Fig. 1 B.

1,1	2,2	3,3
2,3	3,1	1,2
3,2	1,3	2,1

Fig. 2 B.

1	2	3
2	3	1
3	1	2

and

1	2	3
3	1	2
2	3	1

Fig. 1 B.

1	2	3	4		n
2	3	4	5		1
3	4	5	6		2
4	5	6	7		3
n					

Fig. 3.

Inspection of the Latin squares suggests immediately how a Latin square of any order n may be constructed. Construct each row after the first from the preceding now by shifting the symbols one column leftward, the symbol in the first column of the preceding row being transferred to the last column of the new row. (See Fig. 3 A.)

The procedure may be varied by moving the symbols one column to the right instead of to the left, or more generally, "a" columns to the left, where a and n are relatively prime (i.e. have no factor in common). For example, when n is odd, a may be taken equal to 2.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Fig. 4.

Not all Latin squares are of this cyclic type, however. Fig. 4 shows a Latin square of order 4 in which the rows are not "cyclic" permutations of the preceding row.

The existence of Graeco-Latin squares proves to be more difficult to decide for some orders. It is easy to see that there is only one Latin square of order 2 (apart from the choice of symbols), viz.

1	2
2	1

It is therefore clearly impossible to construct a Graeco-Latin square of order 2.

For any odd value of n it is a simple matter to obtain a pair of orthogonal Latin squares both of the cyclic type described above, by taking a equal to 1 and a equal to 2. With a = 2, one obtains Fig. 5, and it is not difficult to see that this is orthogonal to the Latin square in Fig. 3. Hence, there exist Graeco-Latin squares of all odd orders. A little more may be said about orthogonal Latin squares both of cyclic type. It is not very difficult to show (Problem 0) that the cyclic Latin squares constructed as above with $a = a_1$ and $a = a_2$ are orthogonal if and only if the difference $(a_2 - a_1)$ is also relatively prime to n, (as well as a_1 and a_2). If n is an odd prime, we can take a = 1, 2, 3, ..., n - 1, and since all the differences are clearly relatively prime to n, we obtain (n - 1) mutually orthogonal Latin squares of cyclic type.

1	2	3	4	..	n
3	4	5	6		
5	6	7			
7					
n-1					

Fig. 5.

1,1,1,1	2,2,2,2	3,3,3,3	4,4,4,4	5,5,5,5
2,3,4,5	3,4,5,1	4,5,1,2	5,1,2,3	1,2,3,4
3,5,2,4	4,1,3,5	5,2,4,1	1,3,5,2	2,4,1,3
4,2,5,3	5,3,1,4	1,4,2,5	2,5,3,1	3,1,4,2
5,4,3,2	1,5,4,3	2,1,5,4	3,2,1,5	4,3,2,1

Fig. 6.

Fig. 6 shows a set of 4 mutually orthogonal Latin squares of order 5. This is a "best possible" result; it is impossible to have more than (n - 1) mutually orthogonal Latin squares, whether of cyclic type or not. We illustrate the proof by taking n = 5. Suppose there existed 5 mutually orthogonal Latin squares of order 5. By suitable choice of the symbols used for each of the squares, we may assume that when superimposed the first row is as shown in Fig. 7.

1,1,1,1,1	2,2,2,2,2	3,3,3,3,3	4,4,4,4,4	5,5,5,5,5
a,b,c,d,e				

Fig. 7.

Now consider the entries in any other compartment of the square e.g. that in the first column and the second row. None of these entries a, b, c, d and e can equal 1, since there is already a 1 in the first column of each of the five Latin squares. Hence, at least two of them are equal. For the sake of definiteness let us suppose that a = b, and that both are equal to, say, 5. But in this case the first two Latin-squares (i.e. represented by the first two symbols in each compartment) are not orthogonal since the pairing (5, 5) occurs in at least two compartments.

We return to the problem of the existence of Graeco-Latin squares of <u>even</u> order. When n = 4, trial and error soon produces an example, such as Fig. 8. Note that the first of the two orthogonal Latin squares here is that shown in Fig. 4; there is no Graeco-Latin square of order 4 in which either of the Latin squares is of cyclic type. We shall not exhibit a Graeco-Latin square of order 8, but it is not difficult to construct one. These facts, and the following theorem, now enable us to assert that Graeco-Latin squares of order n exist for all values of n which are multiples of 4.

1,1	2,2	3,3	4,4
2,3	1,4	4,1	3,2
3,4	4,3	1,2	2,1
4,2	3,1	2,4	1,3

Fig. 8.

<u>Theorem</u>: If there exist Graeco-Latin squares of order m and n, then there exists a Graeco-Latin square of order mn.

We illustrate the theorem and how it may be proved by taking m = 3, n = 4 and showing how to construct a Graeco-Latin square of order 12. We start with Fig. 2A, and replace each letter by an appropriate Latin square of order 4. In fact we may replace a everywhere by the Latin square in Fig. 4, and α by the second Latin square in Fig. 8, so that aα is replaced by Fig. 8. The symbol b is replaced by Fig. 4 with 5, 6, 7, 8 instead of 1, 2, 3, 4 respectively, and c is replaced by Fig. 4 with 9, 10, 11, 12 instead of 1, 2, 3, 4. Similarly, β and γ are obtained from α by substitution of 5, 6, 7, 8 or 9, 10, 11, 12 for 1, 2, 3, 4. See Fig. 9.

1,1	2,2	3,3	4,4	5,5	6,6	7,7	8,8	9,9	10,10	11,11	12,12
2,3	1,4	4,1	3,2	6,7	5,8	8,5	7,6	10,11	9,12	12,9	11,10
3,4	4,3	1,2	2,1	7,8	8,7	5,6	6,5	11,12	12,11	9,10	10,9
4,2	3,1	2,4	1,3	8,6	7,5	6,8	5,7	12,10	11,9	10,12	9,11
5,9	6,10	7,11	8,12	9,1	10,2	11,3	12,4	1,5	2,6	3,7	4,8
6,11	5,12	8,9	7,10	10,3	9,4	12,1	11,2	2,7	1,8	4,5	3,6
7,12	8,11	5,10	6,9	11,4	12,3	9,2	10,1	3,8	4,7	1,6	2,5
8,10	7,9	6,12	5,11	12,2	11,1	10,4	9,3	4,6	3,5	2,8	1,7
9,5	10,6	11,7	12,8	1,9	2,10	3,11	4,12	5,1	6,2	7,3	8,4
10,7	9,8	12,5	11,6	2,11	1,12	4,9	3,10	6,3	5,4	8,1	7,2
11,8	12,7	9,6	10,5	3,12	4,11	1,10	2,9	7,4	8,3	5,2	6,1
12,6	11,5	10,8	9,7	4,10	3,9	2,12	1,11	8,2	7,1	6,4	5,3

Fig 9

It is a simple matter to convince oneself that a similar process must always yield a Graeco-Latin square of order mn.

So far we have seen how to construct Graeco-Latin squares of order n, where n is odd or any multiple of 4. This only leaves the values 2, 6, 10, 14 ....(i.e. numbers which leave a remainder of 2 on division by 4), and we already know that there is no Graeco-Latin square of order 2. Euler tried hard to find a Graeco-Latin square of order 6, but did not succeed and declared that he had little hesitation in conjecturing that none existed, and he extended his guess to all the other numbers in the list as well. As regards the value n = 6, Euler was eventually proved correct in 1900 by a mathematician named Tarry, who used the not very exciting method of examining all the possibilities. This was tedious and lengthy enough, but a similar programme even for the next smallest case, $n = 10$, would involve a vastly greater amount of labour. Indeed, even using a large computer to search for orthogonal Graeco-Latin squares of order 10 proved to be impracticable. After two hours, the machine found no such square, but the programmer estimated that it would take at least 100 hours before a significant portion of the possibilities had been examined.

Euler, one of the most prolific mathematicians of all time produced an enormous amount of new mathematics that was immediately useful and important, but he did not imagine that his little investigation on Graeco-Latin squares possessed either quality.

However, Sir R. Fisher (one of the most important figures in the twentieth century development of statistical methods) demonstrated in the early 1920's that there was an application of Graeco-Latin squares in connection with the efficient design of experiments.

Suppose, for example, it is desired to compare experimentally five different fertilizers on wheat. It would be possible to plant a field of wheat, and apply the fertilizers in five equal strips. This experiment would give valuable results if one could be certain that the field was equally fertile everywhere, but the results would not be valid if the fertility varied from one strip to the next. To prevent variation in fertility in either direction across the field from making the experiment worthless, Fisher pointed out that the field should be divided into 25 plots and the fertilizers applied as indicated by a Latin square of order 5.

Next, suppose that it is desired to test the fertilizers on 5 different varieties of wheat. Now the most efficient way of designing the experiment makes use of a Graeco-Latin square of order 5. The 5 varieties of wheat are sown as indicated by the positions of the Latin letters, and the five fertilizers are applied by reference to the Greek letters.

Graeco-Latin squares are now widely used for designing experiments in biology, medicine, sociology and even marketing. The Graeco-Latin square is simply the chart of the experiment. Its rows take care of one variable, its columns the care of another, the Latin symbols a third, and the Greek symbols a fourth. Another factor can be introduced if it is possible to find yet another Latin square orthogonal to each of the two in the Graeco-Latin array.

Euler's conjecture was eventually shown to be incorrect as a result of researches on a more general class of objects in combinatorial analysis known as "Balanced Incomplete Block Designs." Bose and Shrikhande noticed that an earlier paper by Parker on this topic threw grave doubt upon the truth of Euler's conjecture, and using its results they were able to construct Graeco-Latin squares of order 22. Then Parker himself made a further contribution which amongst other things produced a Graeco-Latin square of order 10. Eventually it became clear that Graeco-Latin squares of all orders greater than 6 could be constructed. We conclude by exhibiting a Graeco-Latin square of order 10, discovered by Parker.

00	11	22	33	44	55	66	77	88	99
65	30	17	94	52	46	78	89	21	03
42	05	38	57	96	79	81	10	63	24
97	62	09	26	71	80	35	43	14	58
28	47	61	70	85	02	93	34	59	16
19	98	40	82	67	23	04	51	36	75
31	29	95	48	13	64	50	06	72	87
74	86	53	15	20	91	49	68	07	32
56	73	84	69	08	37	12	25	90	41
83	54	76	01	39	18	27	92	45	60