SOLUTIONS OF SENIOR MATHEMATICS COMPETITION QUESTIONS.

Question 1. Let a, b, c, d, e, f represent the integers 1 to 6 in some order.

If $a + b < c + d$ and $c + e < a < f$, identify the values of the six integers and prove that the solution is unique.

ANSWER: Since $c + e \geq 3$ and $f \leq 6$ either $a = 4$ or $a = 5$.

Case I. $a = 5$. Since $c + e \leq 4$ either c or e = 1 $\therefore b \geq 2$.
Also $c \leq 3$. Since $f > a$, $f = 6$ $\therefore d \leq 4$.
$\therefore d + c \leq 4 + 3 = 5 + 2 < a + b$, contradicting $a + b < c + d$.
Hence no solution is possible if $a = 5$.

Case II. $a = 4$. $\therefore c + e = 3$, and e and c are 1 and 2 (not necessarily respectively). Hence $b \geq 3$.
$\therefore 7 \leq a + b \leq c + d - 1 \leq 2 + 6 - 1 = 7$.
The only possibility is for the equals sign to apply throughout. Hence $a = 4$, $b = 3$, $c = 2$, $d = 6$, $e = 1$, $f = 5$ is the only solution.

Question 2. Prove that {1, 2, 3} is the only set of three distinct positive integers with no common factor such that each integer divides the sum of the other two.

ANSWER: Let x, y, z with $x < y < z$ be three positive integers satisfying the requirements.
Since $x + y$ is a multiple of z, but less than 2z, we must have $z = x + y$.

This equation shows that no two of the three integers have a common factor, since such a factor would clearly divide the third integer also, contradicting the data.

ANSWER to Question 2. (cont'd)

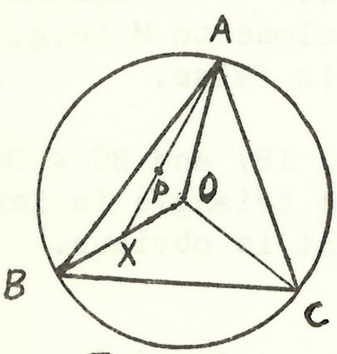
Since y divides $x + z = 2x + y$, 2x is a multiple of y. But since x and y have no factor in common, y is a factor of 2.
$\therefore$ since $y > x \geqslant 1$, we have $y = 2$. Hence $x = 1$, and $z = 3$.

Question 3. If r is the circumradius of a triangle ABC then for each point P within the triangle the smallest of the distances AP, BP, CP is $\leqslant r$.

What can you say about the truth of the statement: "the largest of the distances AP, BP, CP is $\geqslant r$"?

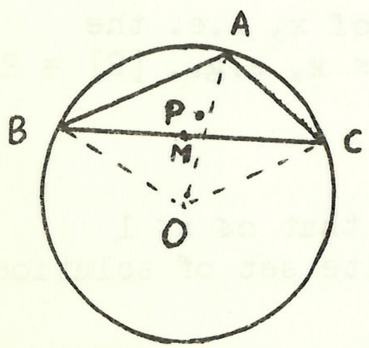
Formulate and prove similar statements concerning the radius of the inscribed circle.

ANSWER: The first statement may be proved as follows:-



I_A

If O is the circumcentre, P lies in one of the regions OAB, OAC, or OBC. Without loss of generality, assume P is in Δ OAB, and $AP \leqslant PB$. If P is on OA and/or OB the result is immediate. Otherwise produce AP to cut OB at X. Then $2AP \leqslant AP + PB < AX + XB < AO + OB = 2r$. (Using the fact that the length of one side of a triangle is less than the sum of the lengths of the other two).
$\therefore AP < r$.



I_B

The second statement is clearly false for the point P in the second figure. It is easy to show that:-

i) The statement is true if Δ ABC is acute angled or right-angled
ii) For an obtuse angled triangle it is always possible to find points for which the statement is false

ANSWER to Question 3. (cont'd)

iii) If the largest angle exceeds 150° the statement is false for all points P within the triangle.

Proof:

(i) Figure I A. If $\angle BCP > \angle BCO$, we have $2r = BO + OC < BP + PC \leqslant 2 \max.(BP, PC)$.
If $\angle BCP < \angle BCO$ we have $\angle ACP > \angle ACO$ and $r \leqslant \max(AP, PC)$.

(ii) Let M be the mid point of the longest side (opp. the obtuse angle A). If $AM \geqslant BM$ then $\angle B > \angle BAM$ and $\angle C > \angle CAM$, whence, adding $\angle B + \angle C > \angle BAC > 1$ rt. angle. This is impossible since $\angle B + \angle C + \angle BAC = 2$ rt. angles, hence we must have $AM < BM$.
$\therefore BM = \max.(BM, AM, CM) = 1/2\ BC < 1/2$ diameter of circumcentre $= r$.
Hence if P is sufficiently close to M (e.g. if $PM < r - BM$) the statement is false.

(iii) If $\angle BAC \geqslant 150^\circ$, $\angle BOC \leqslant 60^\circ$ (Fig. 1B) and $BC \leqslant OC = r$. Since the longest side of the triangle is less than r or equal to r the statement is obvious.

Question 4. How many numbers [x] satisfy the equation $[x] = ax$ where a is a given number?
[x] denotes the "integral part" of x, i.e. the largest integer n such that $n \leqslant x$, e.g., $[2] = 2$, $[\pi] = 3$, $[-\sqrt{2}] = -2$.

ANSWER: (i) If $a = o$, all values of x such that $o \leqslant x < 1$ satisfy the equation, an infinite set of solutions.

ANSWER to Question 4. (cont'd)

(ii) If $a < o$, $x = o$ is the only solution, as x and $[x]$ never have opposite signs.

(iii) If $a = 1$, every integer value of x satisfies the question.

(iv) If $o < a < 1$, $x - [x] < 1 \Rightarrow \frac{[x]}{a} - [x] < 1 \Rightarrow [x] < \frac{a}{1-a}$. Also $x \geqslant o$ (since otherwise $\frac{[x]}{x} > 1$.)

Hence, $[x]$ may be any non-negative integer less than $\frac{a}{1-a}$, of which there are $[\frac{a}{1-a}] + 1$ if $\frac{a}{1-a}$ is not an integer, and $[\frac{a}{1-a}]$ if it is. Since $[x]$ determines x, this gives the number of solutions.

(v) If $a > 1$, $x - [x] < 1 \Rightarrow \frac{[x]}{a} - [x] < 1 \Rightarrow [x] > -\frac{a}{a-1}$.

Also $x < o$ (otherwise $a \leqslant 1$). Hence $[x]$ may be any non-positive integer $> \frac{-a}{a-1}$, of which there are $[\frac{a}{a-1}] + 1$, if $\frac{a}{a-1}$ is not an integer, and $[\frac{a}{a-1}]$ if it is. Again this gives the number of values of x which satisfy the equation.

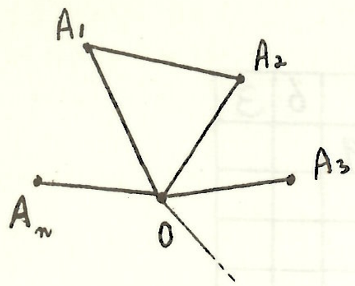
Question 5. At a school dance, every boy dances with at least one girl but no boy dances with all the girls. Similarly, every girl dances with at least one boy but no girl dances with each of the boys.

Prove that there are two boys b_1, b_2 and two girls g_1, g_2 such that b_1 dances with g_1, b_2 with g_2, but neither b_1 with g_2 nor b_2 with g_1.

ANSWER: For b_1, take the boy who has danced with most girls (if there are several who have all danced with a maximum number of girls, select any one of these). For g_2, select any girl who has not danced with b_1; by the data there is at least one. Now take for b_2 any boy with whom g_2 has danced. We are finished if we can find, amongst b_1's partners, a girl g_1 with whom b_2 has not danced. But this must be possible; for if on the contrary b_2 had danced with all b_1's partners and g_2 as well, he would have danced with more girls than b_1, and this is ruled out by our original choice of b_1.

Question 6. From each city of a certain country, a plane takes off and lands at the airport of the nearest city. (You may assume that all distances between airports are different). Prove that there is no airport at which more than 5 planes land.

ANSWER: Suppose planes from A_1, A_2, ..., A_n, all fly to the airport at O. In Δ OA_1A_2, $A_1A_2 > A_1O$ (since otherwise the plane leaving A_1 flies to A_2). Sim. $A_1A_2 > A_2O$. $\therefore$ $\angle A_1OA_2$, being opposite the largest side of the Δ, is greater than 60°. Since this applies to each of the other angles at O, and these total only 360°, clearly $n < 6$.



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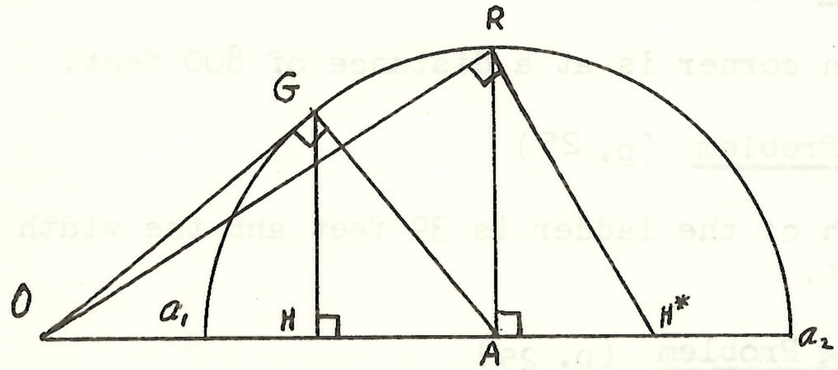
Dr. L.G. Kovacs, of A.N.U., Canberra, has sent the following interesting observation relating to Problem 066.



In Problem 066, put n = 2 and assume, without loss of generality, that $a_1 \leqslant a_2$. Interpret the letters in the diagram as denoting the distance of the relevant point from the point marked O. Check that this interpretation corresponds to that in Problem 066 (adding H* for the antiharmonic mean). Read off that $H \leqslant G \leqslant A \leqslant R \leqslant H^*$, and all these inequalities are strict unless $a_1 = a_2$.

L.G. Kovacs.

ANSWERS:

Re-entrant Queen's Tours (p. 25)

13						12	1
	4				5		
7		10					8
						11	
14	3	9			6		2

2					6	3
8				9		
11						
5						4
12				10	7	1

The numbers indicate in order the squares on which the queen alights. Other solutions are possible.

Mine Shaft (p. 25)

The fourth corner is at a distance of 800 feet.

A Ladder Problem (p. 25)

The length of the ladder is 39 feet and the width of the lane is 51 feet.

A Striking Problem (p. 25)

(1) 12 seconds (2) 13.2 seconds.

A Sporting Proposition (p. 23)

Put one ten dollar note in one hat and all the remaining notes in the other hat. The probability of success is then about 74.4%.

Noughts and Crosses (p. 24)

The first player wins by playing a cross in the centre square and then answering his opponents noughts with symmetrically opposite noughts for one, or at most two, moves.