SOLUTIONS TO PROBLEMS IN PARABOLA, VOL.3, No.2.

JUNIOR

J61 A strip of pasture land is divided into three paddocks whose areas are 35 acres, 10 acres and 65 acres.

On a certain date the owner places 17, 7 and 23 head of cattle (respectively) in the paddocks. The feed in the first is exhausted after 7 weeks, and that in the second after 4 weeks. For how long can the 23 cattle be grazed in the third paddock?

[You will need to assume that initially there is the same amount of food/unit area and that new feed grows at the same constant rate in all three paddocks.]

Answer: Let us take as a unit of food the amount initially available in each acre, i.e. the first paddock has 35 units, the second 10, the third 65. Suppose that food grows at a rate of r units/week/acre, and each cow eats at a rate of c units/week.

Then in the first paddock, after 7 weeks 35 + 7.35r units of food are eaten by 17 cows, so

$$35 + 7.35\,r = 7.17.\,c.$$

From the second paddock we have $10 + 4.10\,r = 4.7\,c$.

If the third paddock is exhausted after n weeks,

$$65 + n.\,65\,r = n.\,23\,c.$$

Solving these equations simultaneously, we find n = 13.

Solution sent by: W. Bundschuh (West Wallsend H.S.)

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Multiplication Puzzle

Replace the letters by digits to make this calculation correct -

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    A B
 x  C D
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    A B
  E A
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  E C B
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(Answer p. 20)

J62 A launch which keeps up a steady speed through the water leaves the boat house and travels upstream. After 2 miles it meets some driftwood floating downstream. The launch travels upstream for another hour and then returns to the boat house, arriving at the same moment as the driftwood. Calculate the speed of the stream.

Answer: The launch maintains a constant speed relative to the stream, so will return to the driftwood exactly 2 hours after meeting it. In this time, the driftwood has travelled 2 miles, so the stream is flowing at 1 mile/hour.

Note that we do not have enough information to find the speed of the launch.

Solution sent by: W. Bundschuh
J. Waterford (St. Joseph's Coll. Hunters Hill)

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J63 Prove that the square of every prime number greater than 3 leaves a remainder of 1 when divided by 12.

Answer: A prime number > 3 cannot leave a remainder of 0,2,3,4, 6,8,9 or 10 when divided by 12. (For example, if it left a remainder of 8, it would be of the form $12k + 8$, which is divisible by 4 so is not prime). Thus such a number is of the form $12k + 1$, or $12k + 5$, or $12k + 7$, or $12k + 11$.

But $(12k + n)^2 = 144k^2 + 24k + n^2$ has the same remainder as n^2 when divided by 12; and since 1,25,49 and 121 all have remainder 1, the conclusion follows.

Solution sent by: W. Bundschuh.

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OPEN

064 In any party of n people we can always find two who have shaken hands with the same number of other people in the party.

Answer: Each person has shaken hands with at most n - 1 people. Thus put the j-th person into category ("box") i ($0 \leqslant i \leqslant n - 1$) if he has shaken hands with exactly i people. There are n categories, and n people; therefore either there are at least two people in the same category, or there is exactly one person in every category. But in the latter case, there is one person who has shaken hands with everyone else, and one person who has not shaken hands with anyone, which is absurd. Therefore there are two people who have shaken hands with the same number of people.

Solutions sent by: P. O'Sullivan (Randwick B.H.S.);
P. Jablon (Telopea Pk. H.S.);
J. Tong (Waverley Coll.);
S. Byrnes (Macquarie B.H.S.);
D. Nash (Parkes H.S.) H. Clarke (Forest High)
C. Cosgrove (Kingsgrove North High)
M. O'Callaghan (St. Joseph's Coll. Hunters Hill)

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065 Let x be any real number and N be any integer $\geqslant 1$. Then there exists a rational number p/q (with p and q integers and $0 < q \leqslant N$) such that the difference $|x - p/q|$ is less than $1/Nq$. (For example, if $x = \pi$ and $N = 10$, we can take $p/q = 22/7$. Then $|\pi - 22/7| < 1/70$).

Answer: For each of the (N + 1) integers q, $0 \leqslant q \leqslant N$, consider $\{qx\} = qx - [qx]$. Here $[qx]$ is the largest integer not greater than qx, so that $\{qx\}$ is the fractional part of qx, $0 \leqslant \{qx\} < 1$. Take N boxes, labelled 1,2,...,N, and put $\{qx\}$ into the box labelled j if

$$\frac{j-1}{N} \leqslant \{qx\} < \frac{j}{N}.$$

By the P.H.P. there is a box containing at least two numbers $\{q_1x\}$ and $\{q_2x\}$ where $0 \leqslant q_1 < q_2 \leqslant N$.

Since they both lie in an interval of length $\frac{1}{N}$ we have

$$|\{q_1x\} - \{q_2x\}| < \frac{1}{N} \text{ i.e. } |(q_2x - [q_2x]) - q_1x - [q_1x]| < \frac{1}{N}.$$

Putting $q = q_2 - q_1$ and $p = [q_2x] - [q_1x]$ this is

$$|qx - p| < \frac{1}{N}; \text{ whence } \left|x - \frac{p}{q}\right| < \frac{1}{Nq}.$$

Solution sent by: P. O'Sullivan, P. Jablon, C. Cosgrove.

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066 If $a_1, a_2, \ldots\ldots, a_n$ are positive numbers we define the arithmetic mean $A = \dfrac{a_1 + a_2 + \ldots\ldots + a_n}{n}$; the geometric mean $G = \sqrt[n]{(a_1 a_2 \ldots . a_n)}$; the harmonic mean $H = \dfrac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}\right)}$, and the root mean square $R = \sqrt{\frac{1}{n}(a_1^2 + a_2^2 + \ldots\ldots + a_n^2)}$. A theorem asserts that $A > G$ regardless of the values of $a_1, \ldots\ldots, a_n$. Assuming this, determine the order of all four quantities, and prove that it is independent of the values of $a., a_n, \ldots . a_n$.

Answer: The harmonic mean H is the reciprocal of the arithmetic mean A' of the numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$; thus $H = \frac{1}{A'}$. Now $A' \geq G'$, the geometric mean of these numbers, and $G' = \frac{1}{G}$, so $H = \frac{1}{A'} \leq \frac{1}{G'} = G \leq A$.

Note that $a_i^2 + a_j^2 \geq 2a_ia_j$. Therefore $\sum_{i \neq j} (a_i^2 + a_j^2) \geq \sum_{i \neq j} 2a_ia_j$ i.e. $(n-1)\sum_{i=1}^{n} a_i^2 \geq \sum_{i \neq j} 2a_ia_j$.

Add $\sum_{i=1}^{n} a_i^2$ to each side: $n\sum_{i=1}^{n} a_i^2 \geq \sum a_i^2 + 2a_ia_j = (\sum a_i)^2$.

Thus $\frac{(\sum a_i^2)}{n} \geq \left(\frac{\sum a_i}{n}\right)^2$ i.e. $R^2 \geq A^2$.

Therefore $R \geq A \geq G \geq H$. (See also p. 31).

Solution sent by: P. O'Sullivan, P. Jablon, J. Tong, D. Nash, H. Clarke, C. Cosgrove.

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067 Given n positive numbers $x_1, x_2, \ldots\ldots, x_n$ prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \ldots\ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

Answer: Consider the set of positive numbers $\frac{x_1}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_n}{x_1}$.

The geometric mean G of these numbers is clearly 1; the inequality merely expresses the fact that the arithmetic mean of these numbers $\geqslant$ G.

Solution sent by: P. O'Sullivan, P. Jablon, J. Tong, C. Cosgrove, S. Byrnes, H. Clarke.

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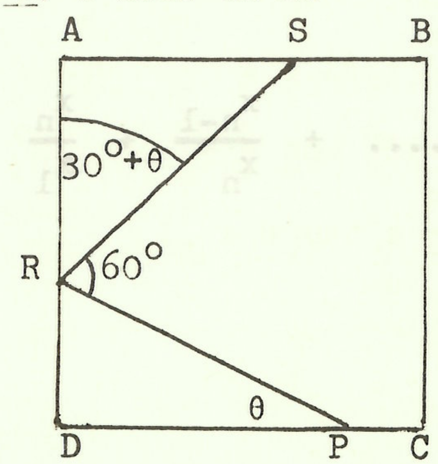
068 Given a square of side 1 unit of length, and a point P on one side of the square, show how to construct, with ruler and compasses only, an equilateral triangle inscribed in the square with one vertex at P. Also find the smallest and largest possible lengths of the side of an equilateral triangle inscribed in the square.

Answer: A construction using trigonometry.
Let P lie on the side CD of the square ABCD. By symmetry, we may assume that DP = a $\geqslant \frac{1}{2}$. One vertex R of the equilateral triangle to be constructed must lie on AD (because the angle between BA and the line joining A to the midpoint of CD is less than 60°). The other vertex S can lie on AB or BC, depending on the value of a.

Case I: S lies on AB



If we can construct the angle θ, or equivalently the length DR, so that the length SR, at angle 60° to PR, is equal to PR, then the triangle PSR is automatically equilateral.

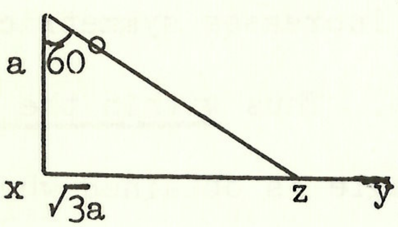
Then $PR = a/\cos\theta$, and $DR = 1 - SR \cos(30^{\circ} + \theta)$.

Thus $DR = 1 - \dfrac{a}{\cos\theta}(\cos 30^{\circ} \cos\theta - \sin 30^{\circ} \sin\theta)$

$$= 1 - \frac{\sqrt{3}}{2} a + \frac{1}{2} a \tan\theta. \quad \text{But } DR = a \tan\theta,$$

so $(1 - \frac{1}{2})\, DR = \dfrac{2-\sqrt{3}}{2} a$, i.e. $DR = 2-\sqrt{3}a$.

Such a length is easy to construct, using the fact that $\tan 60^{\circ} = \sqrt{3}$. Thus a right-angle triangle with one angle 60° and one side equal to the length a produces length $\sqrt{3}a$. If XU = 2, then ZY = $2-\sqrt{3}a$. This length is then measured off along DA with a pair of compasses, to give the point R. SR is constructed at angle 60° to PR, S being the point of intersection with AB.



This construction works when S lies between A and B, rather than on the extended line AB. (The construction still gives an equilateral triangle in the latter case but not inscribed in the square). This is true when $AS \leqslant 1$.

i.e. $SR \sin(30^{\circ} + \theta) \leqslant 1$

$$\frac{a}{\cos}(\sin 30^{\circ} \cos\theta + \cos 30^{\circ} \sin\theta) \leqslant 1$$

$$\frac{1}{2} a + \frac{\sqrt{3}}{2} a \tan\theta \leqslant 1.$$

Now $a \tan\theta = DR = 2 - \sqrt{3}a$.

Thus Case I occurs when $a + \sqrt{3}(2 - \sqrt{3}a) \leqslant 2$,

i.e. $\sqrt{3} - 1 \leqslant a \; (\leqslant 1)$.

The length of side of the triangle is then $\sqrt{a^2 + DR^2} = 2\sqrt{a^2 - \sqrt{3}a + 1}$. The parabola $y = a^2 - \sqrt{3}a + 1$ has a minimum value at the point $a = \frac{\sqrt{3}}{2}$ (i.e. exactly at the midpoint of the range of values of a in Case I.) At this point, the side of the triangle attains its minimum (within Case 1 !) of $2\sqrt{\frac{3}{9} - \frac{3}{2} + 1} = 1$. (S lies vertically above P, and R is the midpoint of AD).

The parabola has no maximum, but increases symmetrically on both sides of its lowest point. Thus <u>within the range</u> $\sqrt{3} - 1 \leqslant a \leqslant 1$, the largest triangle is obtained when $a = 1$ and again when $a = \sqrt{3} - 1$, the length of side being $2\sqrt{2-\sqrt{3}}$ (in both cases, one vertex of the triangle lies in a corner of the square).

(Continued next page)

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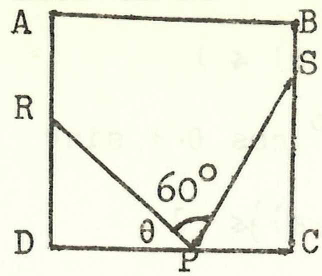
<u>Multiplication Puzzle</u> (p. 13)

<u>Answer</u>. A = 2, B = 3, C = 4, D = 1, E = 9.

Case II: S lies on BC



In this case, we must construct θ (or DR) so that PR = SP where SP makes an angle of 60° with PR; PSR will then automatically be an equilateral triangle.

Again DR = a tan θ, $PR = PS = \frac{a}{\cos\theta}$,

and $a + SP\cos(180^\circ - (60^\circ + \theta)) = 1$

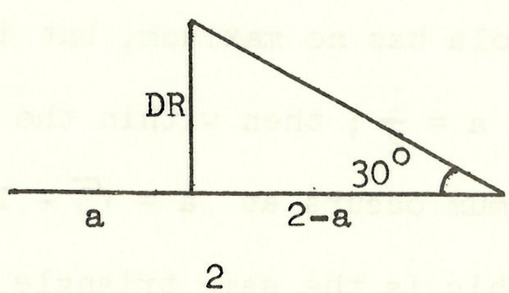
i.e. $a + \frac{a}{\cos\theta}(-\cos 60^\circ \cos\theta + \sin 60^\circ \sin\theta) = 1.$

$$a - \frac{1}{2}a + \frac{\sqrt{3}}{2}\,a\tan\theta = 1.$$

so $\frac{\sqrt{3}}{2}\,DR = 1 - \frac{1}{2}a$

$$DR = \frac{1}{\sqrt{3}}(2-a).$$

This length is also easily constructed



This case occurs when SC $\leqslant 1$

i.e. $SP \sin(60^\circ + \theta) \leqslant 1$

$$\frac{a}{\cos\theta}(\sin 60^\circ \cos\theta + \sin\theta \cos 60^\circ) \leqslant 1$$

i.e. $\frac{\sqrt{3}}{2} a + \frac{1}{2}\left(\frac{1}{\sqrt{3}}(2-a)\right) \leqslant 1$

$$3a + 2 - a \leqslant 2\sqrt{3}$$

$(\frac{1}{2} \leqslant)\ a \leqslant \sqrt{3} - 1$; this is exactly the set of values not covered by Case I.

The length of the side of the triangle is $\sqrt{a^2 + DR^2}$

$$= \sqrt{a^2 + \frac{1}{3}(4 - 4a + a^2)}$$

$$= \sqrt{\frac{4}{3}(a^2 - a + 1)}.$$

The parabola $y = \frac{4}{3}(a^2 - a + 1)$ has a minimum at $a = \frac{1}{2}$; at this point therefore, the minimum triangle within Case II occurs. It has length 1 ; comparing with Case I, we see that 1 is the smallest side of triangle possible, occurring at $a = \frac{1}{2}$ and $a = \frac{\sqrt{3}}{2}$.

Again this parabola has no maximum, but increases as a increases beyond $a = \frac{1}{2}$; then within the range $\frac{1}{2} \leqslant a \leqslant \sqrt{3} - 1$, the maximum occurs at $a = \sqrt{3} - 1$, with length of side $2\sqrt{2-\sqrt{3}}$. This is the same triangle as that obtained by Case I, and is the largest possible equilateral triangle.

Solution sent by: P. O'Sullivan, J. Tong, S. Byrnes, P. Jablon, C. Cosgrove.

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069 A ten pin bowler has eleven weeks to prepare for a tournament. He decides to train by playing at least one game every day, but he cannot afford to play more than twelve games in any one week. Prove that in some number of consecutive days he plays exactly 20 games.

Answer: Suppose the bowler plays x_1 games on the first day, by the end of the second day has played x_2 games in all (so that $x_2 \geqslant x_1 + 1$) and in general, by the end of the n -th day has played x_n games. Thus after 11 weeks, we have the following set of integers $x_1 < x_2 < x_3 < \ldots < x_{77}$.

Let $y_i = x_i + 20$; then also $y_1 < y_2 < \ldots < y_{77}$; altogether we have a set of 154 integers. Now no more than 12 games were played in any one week, so $x_{77} \leqslant 11.12 = 132$; and therefore each of these integers is less than or equal to $132 + 20 = 152$. Thus at least two integers are equal. But no two of the x's are equal, and no two of the y's are equal. Thus

$$x_i = y_j = x_j + 20, \text{ for some } i,j.$$

Then i must be greater than j, and on the i-j consecutive days i-j+1, i-j+2, ..., i-j, the bowler has played a total of $x_j - x_i = 20$ games.

Solution sent by: P. O'Sullivan, J. Tong, S. Byrnes.

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A Sporting Proposition (answer p. 32).

A mathematician confronted his son (who wanted to borrow ten dollars) with the following proposition. He took twenty ten dollar notes, twenty one dollar notes, and two hats. "You may divide these forty notes into two bundles in any way you please", he said. "I will put one bundle (shuffled) in each hat. You will be blindfolded and will choose either hat. Having made your choice you will draw one note from that hat. If it is ten dollars you may keep it. If not, you wash the car".

How should his son divide the money to give himself the best possible chance of getting the ten dollars, and what is his probability of success?

070 A lives in a street in which there are houses numbered from 1 to 30. B wants to know the number of A's house.

B asks A: "Is it greater or less than 15?" A answers, but he lies.

B asks A: "Is it a perfect square?" A answers truthfully.

B says to A: "Is it odd or even? That will fix it definitely."

"Even", says A truthfully. B tells him the number, but he is wrong. What was the number of A's house?

Answer: The housenumber must clearly be a perfect square; for if A had told B it was not a square, then the further information of being odd or even cannot "fix it definitely". The perfect squares less than 30 are 1, 4, 9 (between 1 and 15) and 16, 25.
Suppose A had answered "less than 15" to B's first question; then again the number will not be definitely fixed if A should answer "odd" to B's third question; thus A in fact answered "greater than 15". But he lied; thus the number is an even perfect square $\leqslant 15$. The answer is therefore 4.

Solution sent by: P. O'Sullivan, J. Tong, S. Byrnes, D. Nash, H. Clarke, C. Cosgrove.

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Noughts and Crosses (Answer p.32)

Ordinary noughts and crosses is a draw if both sides play correctly. However if the rules are altered so that each player may add either a nought or a cross when it is his turn to move, the winner being as usual the first to complete a row of three noughts or three crosses, correct play will not produce a draw. Which player ought to win, and what is his strategy?

PUZZLES

Mine Shaft:

A mine shaft sunk vertically on flat land reaches a point exactly 700 feet from one corner of a rectangular plot of farmland, 400 feet from the opposite corner and 100 feet from a third corner. How far is the point from the fourth corner?

(Answer p.32.)

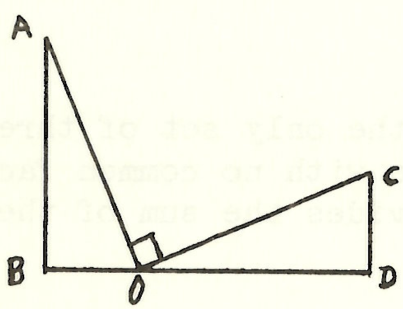
Re-entrant Queen's Tours:

Place a queen on the left hand bottom corner of the chess board and in 14 moves return to this square, having traversed every square on the board at least once. (The queen moves horizontally, vertically or diagonally any desired number of squares).

Find a similar re-entrant queen's tour on a seven-by-seven board in twelve moves.

(Answer p.32).

A Ladder Problem: (Answer p.32)



AB is a building 36 feet high, and CD a house 15 feet high. A ladder OA just reaches the top of the building AB, and if it is turnedthrough a right-angle about O it will just reach the top of the house CD. How long is the ladder and how far apart are the buildings?

A Striking Problem: (Answer p.32)

If a clock takes six seconds to strike six, how long does it take to strike (1) eleven, and (2) twelve?

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