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CONTINUED FRACTIONS

We are all familiar with the fact that every real number can be expressed as a decimal (or has a decimal expansion). Such expansions are of course useful in allowing an easy comparison of the size of numbers, convenient addition, and so forth. The decimal expansion also has the property [*1] that a number α is rational, i.e. of the form p/q with p q integers and $q \neq 0$, if and only if it has a decimal expansion which either terminates or has recurring blocks (repeats). Thus an irrational number, such as $\pi = 3.14159\ldots\ldots$ must have a non-terminating, non-recurring decimal expansion.

Of course we cannot know that π is irrational by looking at its decimal expansion for we could never look far enough to ensure that it never terminated or recurred. Indeed when we write $\pi = 3.14159\ldots\ldots$ all we do is to give a convenient approximation for π

$$\left|\pi - \frac{314159}{100{,}000}\right| < \frac{1}{100{,}000}$$

Actually we give a series of approximations, for we have

$$|\pi - 3| < 1, \quad \left|\pi - \frac{31}{10}\right| < \frac{1}{10}, \quad \left|\pi - \frac{314}{100}\right| < \frac{1}{100}, \ldots\ldots$$

I should remark that these approximations are not very 'economical', for it is well known that

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{70}$$

and here we have a rational approximation with the small denominator 7 within 1/70 of π.

In this article I shall describe the 'continued fraction expansion' of real numbers, the main property of this expansion

[*1]: (see Parabola Vol. 2 No. 3.)

being that it gives very economical rational approximations to numbers; indeed it gives 'best possible' approximations. It is a little difficult to explain precisely what is meant by a 'best possible' approximation so for the moment let us say the following:

p/q (p,q integers and $q \neq 0$) is a very economical approximation for α if

$$|\alpha - p/q| < \frac{1}{q^2}, \quad \text{(or equivalently } |q\alpha - p| < \frac{1}{q}\text{)} .$$

We shall see that there do exist very economical approximations for any irrational number α. Indeed a theorem of Hurwitz states that for any irrational α there exist infinitely many p/q so that

$$|\alpha - p/q| < \frac{1}{\sqrt{5}\, q^2}$$

This is truly a 'best possible' theorem, for if $\sqrt{5}$ is replaced by a larger number it is possible to find irrational numbers α for which it is not true.

By the continued fraction expansion of α we shall mean the expression

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

where $a_0, a_1, a_2, \ldots$ are integers all of which, except perhaps a_0, are positive. For convenience, and to save space, we shall write

$\alpha = [a_0, a_1, a_2, \ldots\ldots\ldots]$ as a brief notation for the complicated expression above.

Also we shall need to denote by $[\alpha]$ the largest integer less than or equal to α. For example

$$[22/7] = 3 \qquad [\sqrt{2}] = 1 \qquad [-\pi] = -4$$

I can now explain the <u>algorithm</u>, or calculating scheme by which we obtain the continued fraction expansion.

Step by step, put

$$[\alpha] = a_o$$

$$\alpha - a_o = \tau_o$$

$$1/\tau_o = \alpha_1$$

$$[\alpha_1] = a_1$$

$$\alpha_1 - a_1 = \tau_1$$

$$1/\tau_1 = \alpha_2$$

and so on ....terminating if at any stage $\tau_k = o$.

Then

$$\alpha = [a_o, a_1, \ldots\ldots.]$$

and in particular if $\tau_k = o$ then $\alpha = [a_o, a_1, \ldots., a_k]$. To see that the algorithm works, notice that

$$\alpha_k - a_k = \alpha_k - [\alpha_k] = \tau_k \quad \text{so that} \quad 0 \leqslant \tau_k < 1$$

Hence

$$1/\tau_k = \alpha_{k+1} > 1 \text{ and thus } a_{k+1} = [\alpha_{k+1}] \geqslant 1.$$

Next observe that

$$\alpha = a_o + \tau_o = a_o + \frac{1}{\alpha 1} = a_o + \frac{1}{a_1 + \tau_1} = a_o + \frac{1}{a_1 + \frac{1}{\alpha_2}} = \ldots..$$

Thus we see that not only does the algorithm work, but since it gives a unique a_k at each step, the continued fraction expansion of α is essentially unique.

Problem 1 (a) Show that if the continued fraction of α terminates then α is rational

(b) Use the algorithm to show that if α is rational then its continued fraction terminates.
[at each step put $\alpha_k = r_k/s_k$ r_k, s_k relatively prime integers.

Then $r_k/s_k = a_k + s_{k+1}/s_k$ with $0 \leqslant s_{k+1} < s_k$ and $r_{k+1} = s_k$.

Hence we obtain a strictly decreasing sequence of integers

$s > s_1 > s_2 > \ldots\ldots \geqslant 0$ which must terminate with 0.]

Given a non-terminating (or, for that matter, a long terminating) continued fraction

$$\alpha = [a_o, a_1, a_2, \ldots\ldots]$$

Consider this continued fraction 'cut' at k. i.e. $[a_o, a_1, \ldots, a_k]$. By problem 1 (a) such a cut-off fraction represents a rational number p_k/q_k called the k-th convergent of α.

Now it turns out that the convergents can be easily calculated from the recursive formulae

$$p_o = a_o \quad p_1 = a_1 a_o + 1 \quad p_k = a_k p_{k-1} + p_{k-2} \qquad (k = 2, 3, \ldots.)$$
$$q_o = 1 \quad q_1 = a_1 \quad q_k = a_k q_{k-1} + q_{k-2}$$

Problem 2 Prove these formulae.

[Use induction on k; then $[a_o, a_1, \ldots, a_k, a_{k+1}] = [a_o, a_1, \ldots, a_k + \frac{1}{a_{k+1}}]$

$$= \frac{(a_k+1/a_{k+1})p_{k-1}+p_{k-2}}{(a_k+1/a_{k+1})q_{k-1}+q_{k-2}} = \frac{a_{k+1}(a_k p_{k-1}+p_{k-2})+p_{k-1}}{a_{k+1}(a_k q_{k-1}+q_{k-2})+q_{k-1}}$$

$$= \frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k+1}} = \frac{p_{k+1}}{q_{k+1}} \;]$$

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From these formulae we obtain a remarkable identity:

$$p_k q_{k-1} - p_{k-1} q_k = (a_k p_{k-1}+p_{k-2})q_{k-1} - p_{k-1}(a_k q_{k-1}+q_{k-2})$$

$$= -(p_{k-1}q_{k-2} - p_{k-2}q_{k-1})$$

and hence, repeating with k-1, k-2, ..., 2 for k...

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}(p_1 q_o - p_o q_1) = (-1)^{k-1} .$$

In a quite exact way we can write (convince yourself)

$$\alpha = [a_o, a_1, \ldots, a_k, \alpha_{k+1}]$$

and then

$$\alpha = \frac{\alpha_{k+1}p_k + p_{k-1}}{\alpha_{k+1}q^k + q_{k-1}}$$

Hence

$$\alpha - \frac{p_k}{q_k} = -\frac{p_k q_{k-1} - p_{k-1} q_k}{q_k(\alpha_{k+1}q_k + q_{k-1})} = \frac{(-1)^k}{q_k(\alpha_{k+1}q_k + q_{k-1})}$$

Since $\alpha_{k+1} \geqslant a_{k+1} > 0$ certainly $\alpha_{k+1}q_k + q_{k-1} \geqslant q_{k+1} > q_k$

and thus

$$|\alpha - p_k/q_k| < \frac{1}{q_k^2}$$

Hence when α is irrational and has a non-terminating continued-fraction, we obtain infinitely many very economical rational approximations p_k/q_k to α. In fact it turns out that precisely the best possible q_k are picked out by the continued fraction.

As a bonus the continued fraction algorithm does more than just give good approximations. We notice that a rational number $\alpha = p/q$ is precisely the sort of number which satisfies a linear equation $qx - p = 0$ with integer coefficients; the next simplest type of number is a quadratic irrationality satisfying an irreducible (unfactorisable) quadratic equation $rx^2 + sx + t = 0$ with integer coefficients. Remarkably the continued fraction of α is periodic (recurring) whenever, and only if, α is a quadratic irrationality. For example

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = [1,2,2,2,\ldots\ldots] = [1,\dot{2}]$$

$$\sqrt{3} = [1,1,2,1,2,1,2,\ldots\ldots\ldots] = [1,\dot{1},\dot{2}]$$

Problem 3 Try to prove that every periodic continued fraction represents a quadratic irrationality (not very difficult).

The converse to problem 3 is however quite involved and was first proven by Lagrange. One might quite well ask whether there are generalisations of the continued-fraction algorithm which, say, are periodic for n-th degree irrationalities, terminating for lower degree irrationalities; there is in fact a candidate for this property, an algorithm developed by Jacobi and Perron, but whilst the analogue to problem 3 has long been known to be true for it, the analogue to the converse as yet remains a conjecture. I regret to say that I am among many researchers who during the last sixty years have failed to either prove or disprove this conjecture.

Finally I should emphasise that I have by no means described everything interesting about the continued fraction. Those ambitious to know more should attempt to read one or more of the following

Hardy G.H. and Wright E.M. 'An Introduction to the Theory of Numbers' Chapter X 129-153.

LeVeque W.J. 'Topics in Number Theory' Vol. 1, Chapter 9 159-193.

In a future article on 'Algebraic and Transcendental Numbers' I will discuss the significance of economical rational approximations in deciding whether a number is transcendental.

A.J. van der Poorten.

Problem 4. Show that if α is irrational then the convergents p_k/q_k do indeed converge to α i.e. $\lim_{k\to\infty} p_k/q_k = \alpha$.

[use $q_{k+1} > q_k$]

Problem 5 Calculate a few convergents of 3.14159

[3/1, 22/7, 333/106 .... For π = 3.14159 26535 89793 .....

the following approximations occur historically:

3 (The Bible (I Kings VII 23)); $(\frac{16}{9})^2$ = 3.1604..(Rhind Papyrus);

3 10/71 < π <3 10/70 (Archimedes) 287-212 B.C.));

$\frac{355}{113}$ = 3.1415929 ....(Tsu-Chung-Chich (b.A.D.430))]

Problem 6 Show that the even convergents p_0/q_0, p_2/q_2,.... are less than α and the odd convergents greater than α.

[use $\alpha - p_k/q_k = (-1)^k/q_k(\alpha_{k+1}q_k + q_{k-1})$]

Problem 7 Show that in a convergent p_k/q_k the numbers p_k, q_k are relatively prime (have no common factor)

[use $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$]

Problem 8 Work out the simplest continued fraction, that of

$$\frac{\sqrt{5}+1}{2} = \alpha.$$

Problem 9 Show that if α is rational then it has only finitely many very economical approximations p/q, i.e. so that

$$|\alpha - p/q| < \frac{1}{q^2}$$

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The Three Applicants.

An employer determines which to choose of three bright applicants for a position by subjecting them to the following test. He attaches a black marker to the forehead of each and seats them where they can see the other two applicants. They are then told that each forehead bears either a white or a black marker and that at least one black marker has been used. The first person who is sure of the colour of his own marker will be awarded the position (provided his reasons for being sure are sound).

After a couple of minutes one of the applicants correctly announced that his marker was black, and after explaining his reasoning was awarded the position. What was his explanation?

(Answer page 24).