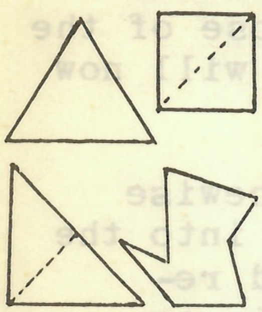
On Dissections of Polygons



To grasp the nature of the contents of this article, consider the four figures in the diagram. One is an equilateral triangle, one a square, one a right angled isosceles triangle and the fourth some other polygon. Is it possible to start with two of these figures, dissect one into pieces by making a number of straight cuts, and reassemble these pieces in a different way to produce the other figures? It is immediately clear that this is impossible if the two figures involved are not of equal area, so we shall assume to start with that the areas are all equal.

It does not take very long to discover the dissections of the square and the right angled triangle indicated by the dotted lines, which enables us to make an affirmative answer to the question under discussion for these two figures. However, even if you have excellent geometrical intuition, it will take you a lot longer to decide whether the equilateral triangle can be dissected into a finite number of pieces which reassemble to form the square. In fact, it is one of the better known discoveries of England's greatest inventor of puzzles H.E. Dudeney, that the equilateral triangle can be cut into as few as four pieces which can then be reassembled to form the square.

This leaves the irregular polygon. You could be forgiven for supposing that no answer could be given to our question involving the polygon and say the square, until precise information concerning the polygon was presented. But you would be wrong. It is an at first sight surprising theorem that any polygon can be transformed into any other polygon of the same area by cutting it into a finite number of polygonal pieces and re-assembling. This was first proved by David Hilbert, one of the greatest German mathematicians of the first half of this century and the close of the last. The steps in a proof of this result will now be presented.

We shall say that two polygons are piecewise congruent if it is possible to transform one into the other by dissection into polygonal pieces and re-assembling. This relation between polygons is an example of what mathematicians call an equivalence relation, because of the following three properties it possesses.

(i) It is a "reflexive" relation:- i.e. a polygon is piecewise congruent to itself.

(ii) It is a "symmetric" relation:- i.e. if polygon A is piecewise congruent to polygon B, then polygon B is piecewise congruent to polygon A.

(These first two properties of an equivalence relation are quite obvious for piecewise congruence)

(iii) It is a "transitive" relation:- i.e. if polygon A is piecewise congruent to polygon B, and polygon B is piecewise congruent to polygon C, then polygon A is piecewise congruent to polygon C.

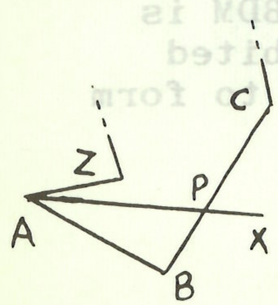
This is not quite so obvious, and as it is needed in the proof of Hilbert's theorem, we shall attempt to make its truth clearer. A is first dissected into pieces which are rearranged to give B. Suppose that these pieces are now weakly re-connected, and then B is dissected into pieces which are rearranged to give C. If the weak re-connections are now broken again it is clear that we have C dissected into pieces which can be rearranged to give A.

Because of the transitivity of piecewise congruence, to prove Hilbert's theorem it is plainly sufficient to show that any polygon is piecewise congruent to a rectangle of the same area whose base is one unit long. To accomplish this there are two main steps:- first, every polygon can be dissected into a finite number of triangles; and second, every triangle is piecewise congruent to a rectangle with base one unit long. We consider these in turn.

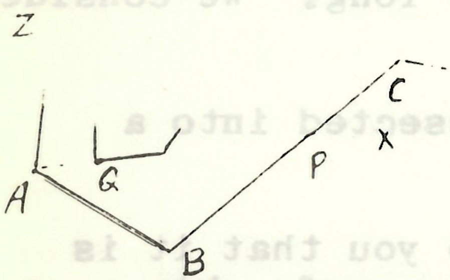
Lemma:- Every polygon can be dissected into a finite number of triangles.

If this seems so obvious to you that it is waste of time reading through the proof, skip down to the next lemma. It is indeed obvious for convex polygons since the diagonals from one vertex accomplish the dissection. A proof for any polygon could proceed by induction on the number of vertices, n.

The result is trivial if n is 3 . Assume that it has been proved for all polygons with fewer than k vertices and consider any polygon with k vertices. Since not all the angles can be re-entrant we can select three adjacent vertices on the perimeter such that the interior angle ABC is less than two right angles.

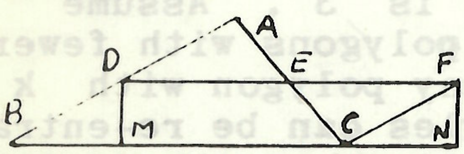


If AC lies entirely within the polygon, we are finished, since after cutting off the triangle ABC, we have a polygon with (k-1) vertices, and by the induction hypothesis this can be dissected into a finite number of triangles. If AC does not lie entirely within the polygon rotate a ray AX, initially coincident with AB, towards AC. Let P be the intersection of AX with BC and stop rotating the ray as soon as AP no longer consists entirely of interior points of the polygon (apart from the boundary points A and P themselves). There are two possibilities: (i) AX is now coincident with AZ, the other side of the polygon sharing the vertex A. In this case we are finished after cutting off the triangle BAZ because of the induction hypothesis: (ii) AX has intersected the boundary of the polygon at one (or more) points between A and P. Let the closest such point to A be Q. Then Q is a vertex (why?) and the diagonal AQ dissects the polygon into two polygons each having fewer than k vertices which can each be further dissected into triangles by the induction hypothesis. This completes the proof of the Lemma.

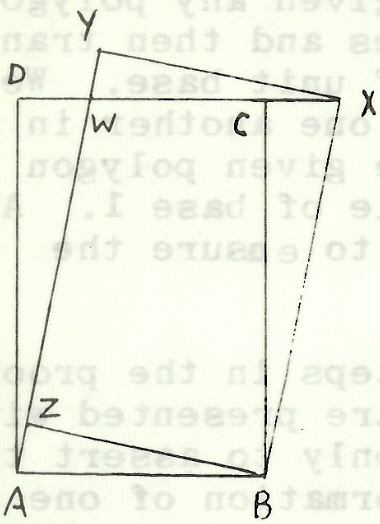


Lemma. Every triangle is piecewise congruent to a rectangle.

Proof. Let BC be the longest side of $\triangle ABC$ and let D and E be mid-points of the sides AB, AC. DE is produced to F so that DE=EF and perpendiculars DM and FN are dropped to BC (produced). It is a simple matter to observe that $\triangle AED$ is congruent to $\triangle CEF$ and $\triangle BDM$ is congruent to $\triangle CFN$. We have thus exhibited a dissection of $\triangle ABC$ which reassembles to form the rectangle DMNF.



Lemma. Any rectangle is piecewise congruent to a rectangle, the length of whose base is a rational number of units.



Proof. If either the length *AD or the breadth *AB of the given rectangle ABCD is rational there is nothing left to prove. In any case it is possible to find a rational number r such that *AD < r < *AC. Let the circle of centre B, radius r, cut DC produced at X (refer to the diagram). Join BX, and draw AY parallel to BX. AY intersects DC between D and C because of our choice of r. Perpendiculars BZ and XY are dropped to the line AY from B and X. It is evident that $\triangle$AZB is congruent to $\triangle$WYX and that $\triangle$ADW is congruent to $\triangle$BCX, so the diagram exhibits a dissection of ABCD which reassembles to form the rectangle BXYZ with the base BX of rational length.

Lemma. A rectangle whose base is of rational length is piecewise congruent to a rectangle whose base is of length one unit.

Proof. Let the length of the base of the given rectangle be m/n units of length, where m and n are integers. This rectangle can obviously be dissected into m rectangles all with the base 1/n units long, and these may be stacked one on top of the other to form a single rectangle. This tall thin rectangle may be subdivided by drawing horizontal lines into n congruent rectangles which if stacked side by side will produce a rectangle of base 1 unit.

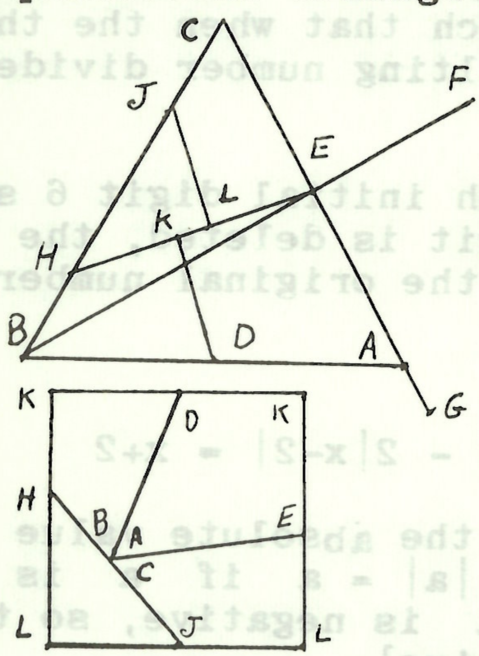
Combining these last three lemmas we see that any triangle is piecewise congruent to a rectangle of unit base, because of the transitivity of piecewise congruence. Now suppose we are given any polygon. We first dissect it into triangles and then transform each of these into a rectangle of unit base. We stack these rectangles on top of one another in a column, and thus observe that the given polygon is piecewise congruent to a rectangle of base 1. As was noted earlier, this suffices to ensure the truth of Hilbert's theorem.

You will notice that the steps in the proof of the theorem enable us, if we are presented with two polygons of equal area, not only to assert that there exists a dissection transformation of one into the other, but actually to find such a dissection. It is clear that there are always many different dissections which enable us to accomplish the transformation, since, for example, from a given dissection we could simply cut one of the pieces in two to obtain a different dissection. The question naturally arises of discovering from amongst all the possible dissection transformations that which is most elegant in that it uses the smallest possible number of pieces.

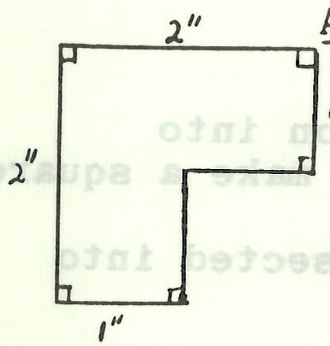
No general process for answering this question is known, and even when both polygons are regular and with few sides the answer is seldom known. For example, it was believed for many years that any transformation of the regular pentagon into a square required a dissection into at least 7 pieces. This belief was shattered when a dissection which used only 6 pieces was discovered, (again by Dudeney, who seems to have had a rare talent for this difficult geometrical pastime). It is even a possibility, though perhaps unlikely, that this result could be improved, as I am not aware that Dudeney's dissection has been proved the best possible.

The analogous problem in three dimensions would consist in transforming one given polyhedron into another of equal volume by dissecting it into a finite number of polyhedral pieces (i.e. using only plane cuts) and rearranging. However Hilbert's theorem proves to have no three dimensional counterpart, i.e. it is possible to find two polyhedra of the same volume which are not piecewise congruent.

In conclusion, Dudeney's dissection of the equilateral triangle to form a square is presented. Let D and E be mid points of the sides AB, AC of the equilateral triangle ABC. Produce BE to F so that *EF=*AE. Let EA produced cut the semicircle with diameter BF at G and let the circle of centre E, radius *EG cut BC at H. Construct the point J on HC such that *HJ=*AE. Join EH and drop perpendiculars DK and JL from D and J to EH. The four pieces AEKD, BHKD, HJL and JCEL can now be reassembled to form a square as indicated in the accompanying diagram.



A Dissection Puzzle



Dissect the polygon in the figure into four pieces which are all congruent to one another

(Answer p. 32)