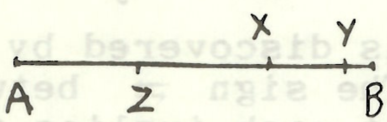
Parabola Volume 4, Issue 2 (1967)

SOLUTIONS OF THE JUNIOR DIVISION COMPETITION PROBLEMS.

Question 1. Two runners start on a straight one mile race track. The slower runner S is given a handicap and starts first. The faster runner F then starts and after 4 minutes of running he passes S. After a further 2 minutes, in which time the faster runner has run to the end of the track and immediately turned back, the runners pass each other again (going in opposite directions). After a further 7 minutes, in which time each has run to an end of the track (opposite ends) and immediately turned round, they meet once again.

Assuming that their speeds are constant, how long does it take each of them to run one mile?

Answer. Let A be the start of the track, B the other end, and X,Y,Z the three points at which the runners pass. In the 7 minutes between the meetings at Y and Z, F runs the line intervals YA and AZ and S the line intervals YB and BZ. Hence the combined distance travelled by the two runners in 7 minutes is (*AY + *YB) + (*AZ + *ZB) = *AB + *AB = 2 miles.



Since their speeds are constant it follows that in the two minutes between the passing at X and the meeting at Y the combined distance travelled is 2/7 of 2 miles = 4/7 miles. But in this time, F has run a distance *XB + *BY and S a distance *XY. Hence *XY + *BY + *XB = 2*XB = 4/7 miles.

Thus *XB = 2/7 miles and *AX = *AB - *XB = 1 - 2/7 = 5/7 miles. F takes 4 minutes to cover the distance *AX of 5/7 miles so his time for the mile is 7/5 of 4 minutes = 5 3/5 minutes. (1)

In 7 minutes F travels 7/4 × 5/7 = 1 1/4 miles and since the combined distance travelled in this time is 2 miles S covers 3/4 mile in 7 minutes. Therefore S takes 4/3 × 7 = 9 1/3 minutes to run one mile. (2)

The required answers are statements (1) and (2).

Question 2. If x and y are two positive numbers whose sum is one show that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9 .$$

Answer. Since squares (of real numbers) are never negative

$$2(1 - 2x)^2 \geq 0$$

$$2 - 8x + 8x^2 \geq 0 \rightleftarrows 2 \geq 8x(1-x) \rightleftarrows 2 \geq 8xy$$

(since $y = 1-x$)

$$\rightleftarrows 2 + xy \geq 9xy \rightleftarrows \frac{1+(x+y)+xy}{xy} \geq 9$$

$$\rightleftarrows \frac{(1+x)}{x}\ \frac{(1+y)}{y} \geq 9 \rightleftarrows (1+\frac{1}{x})\ (1+\frac{1}{y}) \geq 9$$

(Of course, this proof is discovered by first working it backwards. The sign $\rightleftarrows$ between two statements means that each implies the other).

Question 3. Let a and b be two rational numbers. If both ab and $a + b$ are whole numbers, show that a and b are both whole numbers. (A rational number is a number which may be written in the form $\pm m/n$ where m and n are whole numbers.)

Answer Let $ab = h$ and $a + b = k$ where h and k are integers. Eliminating b from the two equations gives $a^2 - ak + h = 0$.

Substituting $a = m/n$ where m and n are integers $(n > 0)$ with no common factor greater than 1, we obtain, (after multiplying by n^2)

$$m^2 - k\,m\,n + h\,n^2 = 0$$

$$m^2 = n\,(km - hn)$$

If p is any prime factor of n , it is a factor of the R.H.S. of this equation, therefore of m^2, and, finally, of m . Hence it is a common factor of m and n. Since m and n have no common factor greater than 1, it follows that there can be no prime factors of n, i.e. $n = 1$. Hence $a = m/n = m$, a whole number, as was to be proved. Then $b = k - a$ is also a whole number.

Alternatively, if $a = m/n$, (in lowest terms), $b = k - a = (kn - m)/n$. This is also in lowest terms since any common factor of n and $(kn - m)$ would also be a factor of $kn - (kn-m) = m$, whereas it is assumed that m and n are relatively prime.

But then $a\,b = \dfrac{m.(kn-m)}{n^2}$ is also in lowest terms. Since this is a whole number we must have $n^2 = 1$ whence $n = 1$ and $a = m$, an integer.

Question 4. Find a whole number N so large that

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} > 10 \, .$$

Answer. Put $S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{N}$. If $2^{\ell} < N$

$$S_N = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \dots \left(\frac{1}{2^{\ell-1}+1} + \dots + \frac{1}{2^{\ell}}\right)$$

$$+ \left(\frac{1}{2^{\ell}+1} + \dots + \frac{1}{N}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right)$$

$$+ \dots + \left(\frac{1}{2^{\ell}} + \frac{1}{2^{\ell}} + \dots + \frac{1}{2^{\ell}}\right) + \left(\frac{1}{2^{\ell}+1} + \dots + \frac{1}{N}\right)$$

$$+ 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots + \frac{2^{\ell-1}}{2^{\ell}} + \left(\frac{1}{2^{\ell}+1} + \dots + \frac{1}{N}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} + \left(\frac{1}{2^{\ell}+1} + \dots + \frac{1}{N}\right)$$

$$= 1 + \frac{\ell}{2} + \left(\frac{1}{2^{\ell}+1} + \dots + \frac{1}{N}\right)$$

> 10 if $\ell = 18$.

Hence it is surely sufficient to take for N any number greater than $2^{18} = 262,144$. (In fact, a more complicated analysis shows that considerably smaller values of N will suffice. The smallest allowable value of N is of the order 12.4×10^3).

Question 5. Two people play a game which consists in their alternatively removing matches from a pile. Each player, when it is his turn, must remove at least one match, but fewer than four times the number remaining in the pile at the completion of his move. (For example, if at some stage there are 35 matches in the pile the next player can remove any number from 1 to 27 inclusive. The game stops when the pile has been reduced to one match, since no legal move is possible.) The winner is the last player to make a legal move. If there are initially 60 matches in the pile, which player ought to win, the first to move or the second? What is his winning strategy?

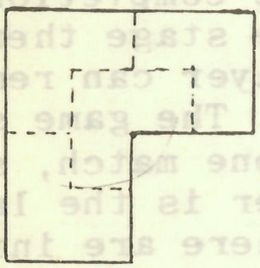
Answer. The first player, A can win by taking 35 matches, leaving $25 = 5^2$ at his first move. If B answers by taking x matches, $1 \leq x < \frac{4}{5}.25$, then A takes 20-x matches leaving 5 after his second turn. B must then take any number, y, between 1 and 3 inclusive, and A wins the game by taking 4-y matches on his third turn, leaving only one match.

In general if A is able to leave 5^n matches after a turn, then B is forced to take fewer than 4.5^{n-1} matches in reply. If he takes k matches, then A takes $(4.5^{n-1}-k)$ matches reducing the pile to 5^{n-1} matches. Eventually A's last move reduces the number of matches to $5^0 = 1$, thus winning the game.

Answers

Level Crossings (p. 22) Forty four

A Dissection Puzzle (p. 7)



The answer is given by the dotted lines in the accompanying figure.

Hats(p.25)E announces that his hat is black. He argues that A must have seen at least one black hat since otherwise he would know that his own was black. Then B, if he had seen three white hats would have known that his own was not white. But he did not know this, so he must have seen at least one black hat. Similarly C, and finally D must have seen a black hat. [This is in essentially the same logical puzzle as appeared in the previous issue of Parabola under the heading The Three Applicants] .

Lost Digits (p.24)

(i) $\sqrt{101124} = 318$

(ii) $110,768 \div 112 = 989$