SOLUTIONS TO PROBLEMS IN PARABOLA, VOL. 4. No. 1.

JUNIOR

J81 The lengths of the sides of a right angled triangle are natural numbers. The length of one side is 14. Find the lengths of the other sides.

Answer. There are no such triangles having the length of the hypotenuse equal to 14, since this would imply the existence of smaller integers x, y such that $x^2 + y^2 = 14^2$. However, if $y \geq x$ then $y^2 \geq x^2$ and $2y^2 \geq 196$, whence $10 \leq y < 14$; and it is a simple matter to check that none of $14^2 - 10^2$, $14^2 - 11^2$, $14^2 - 12^2$, and $14^2 - 13^2$ is a perfect square.

If the side of length 14 is an arm of the right angle, let integers y, x be the lengths of the hypotenuse, and the remaining side respectively. Then $14^2 = y^2 - x^2 = (y-x)(y+x)$. (1) It is impossible for x and y to be of opposite parity (i.e. one odd, the other even) since the difference of their squares is an even number. Hence they are either both even or both odd, and in any case $(y-x)$ and $(y+x)$ are even numbers. Set $y-x = 2h$, and $y+x = 2k$ where h and k are integers and substitute in (1) to obtain $49 = h\,k$. The only possible factorisation with $h < k$ is $h = 1$, $k = 49$. Hence $y-x = 2$, $y+x = 98$, from which we obtain $x = 48$ $y = 50$ as the only solution.

Correct answers received from: R. Sebesfi (St. Joseph's Coll. Hunters Hill), N. Wormald (Syd. Tech. H.S.), A. Pickering (TelopeaPk. H.S.), S. Bashford (Wallsend H.S.).

[1]82 Solve the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{x} = \frac{1}{a+b+x}$.

[A]nswer Rearranging $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b+x} - \frac{1}{x}$

$$\frac{a+b}{ab} = \frac{-(a+b)}{x(a+b+x)}$$

If $a+b=0$, the equation is satisfied by all values of x (except $x=0$, for which it becomes meaningless).

If $a+b\neq0$, division by $a+b$ and cross multiplication gives

$$x^2+(a+b)x + ab = 0$$

$$(x+a)\ (x+b) = 0$$

Hence $x = -a$ or $x = -b$.

Correct answers received from: R. Wildman (Keira B.H.S.) N. Wormald (Syd. Tech. H.S.), A. Pickering (Telopa Pk. H.S.).

[1]83 Show that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are never three terms in the same arithmetical progression.

[A]nswer. Suppose on the contrary that $\sqrt{3}$ is the mth term and $\sqrt{5}$ the nth term after $\sqrt{2}$ in an A.P. whose common difference is d. i.e. $\sqrt{3} = \sqrt{2} + m.d$ and $\sqrt{5} = \sqrt{2} + n.d$, giving $\sqrt{3}\,n - \sqrt{5}\,m = \sqrt{2}\,(n-m)$.

Squaring and rearranging we have

$$\sqrt{15} = \frac{3n^2+5m^2- 2(n-m)^2}{2mn}$$

where both numerator and denominator are positive integers. The proof will be complete if we show that in fact there are no positive integers p,q such that $\sqrt{15} = p/q$. If such do exist, we may assume that they have no common factor, since this may first be cancelled out. Since $p^2 = 15q^2$, 3 is a factor of p^2 and hence of p. Let $p = 3P$ where P is an integer.

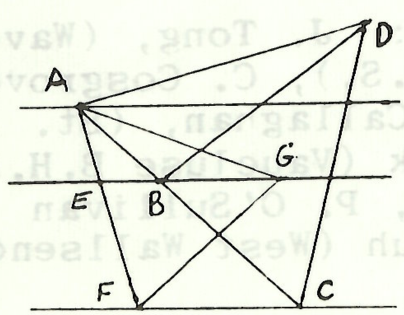
Then $p^2 = 9P^2 = 15q^2$ and $3P^2 = 5q^2$.

Now 3 is a factor of $5q^2$ and therefore of q. Since both p and q have the factor 3, it has proved impossible to find relatively prime integers p and q whose ratio is $\sqrt{15}$.

Correct answers received from. H. Clarke (Forest H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), P. O'Sullivan (Randwick B.H.S.), P. Jablon (Telopea Pk. H.S.), J. Tong (Waverley Coll.) C.W. Evers (Vaucluse B.H.S.), G. Payne (Balgowlah B.H.S.), D. Nash (Parkes H.S.).

O84 Show how to construct, using straight edge and compasses only, an equilateral triangle whose vertices lie on three given parallel straight lines.

Answer Construction:- Draw any transversal ABC and construct the equilateral triangle ACD. Join BD.



Construct transversal AEF such that $\angle CAF = \angle DBX$ (see figure), and construct AG such that $\angle GAF = 60^o$. Join GF. Proof:- $\triangle AEG$ and $\triangle ABD$ are equiangular, since $\angle EAG = \angle BAD = 60^o$ by construction and $\angle AEG = \angle ABG - \angle EAB = \angle ABG - \angle GBD = \angle ABD$.

Hence $\frac{BD}{EG} = \frac{AB}{AE} = \frac{BC}{EF}$. Since also $\angle FEG = \angle CBD$

(supplements of equal angles) we have $\triangle FEG$ similar to $\triangle CBD$, whence $\angle AFG = \angle ACD = 60^o$ and $\triangle AFG$ is equilateral.

Correct answers received from: J. Tong (Waverley Coll.), F. Sharpe (Epping B.H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), K. Louie (Cleveland B.H.S.), D. Hermann (Oak Flats H.S.). P. O'Sullivan (Randwick B.H.S.)

085 A man runs up a certain down-moving escalator and reaches the top after taking 125 steps. He then walks down the escalator slowly, taking one step in the time it took to take 5 steps on the previous trip. He reaches the bottom after 50 steps. How many steps would be visible if the escalator stopped running?

Answer. Let n be the number of steps visible when the elevator is stopped and let k be the unit of time the man takes to walk down one step. Since he takes 50 steps to walk down the down-moving escalator, $(n-50)$ steps have gone out of sight in $50\,k$ units of time. However, the man takes 125 steps to run up the same escalator in 125/5 or 25 k units of time.

Therefore $\dfrac{n-50}{50k} = \dfrac{125-n}{25k}$

whence $n = 100$.

Correct answers received from: J. Tong, (Waverely Coll.), F. Sharpe (Epping B.H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), T. O'Callaghan, (St. Joseph's Coll. Hunters Hill), D. Pollak (Vaucluse B.H.S.), P. Dwyer (St. Aloysius Coll.), P. O'Sullivan (Randwick B.H.S.) W. Bundschuh (West Wallsend H.S.).

086 Three men A, B and C decide to fight a pistol duel along the following lines. They will first draw lots to determine who fires first, second and third. After positioning themselves at the vertices of an equilateral triangle, they will fire single shots in turn and continue in the same cyclic order until two of them have been hit. The man whose turn it is to fire may aim wherever he pleases. Once a man has been hit, whether killed or not, he takes no further part in the duel. All three men know that A always hits what he aims at, B is 80% accurate, and C is 50% accurate.

Assuming that all three adopt the best strategy, and that no one is hit by a shot not aimed at him, who has the best chance to escape unscathed? What is the exact probability of escape of each of the three men?

Answer. Clearly neither A nor B will aim at C while all three are still alive, since his killing would be very probably avenged by the very next shot. Hence each of A and B will aim at the other until one has been hit. It follows that C's best strategy is to fire into the air until one of the others has been hit. Then, of course, he must attempt to hit the remaining contestant.

The course of the duel must follow one of the following three patterns.

Case 1. A fires before B. His first shot hits B. Then C has a 50% chance of hitting A. If he misses, A's next shot finishes the duel. Survival chances, if case 1 applies, are $\frac{1}{2}$, 0 and $\frac{1}{2}$ respectively for A, B and C.

Case 2. B fires before A, but his first shot (aimed at A) misses. The future course of the duel is exactly as described in Case 1. Survival chances, if case 2 applies, are again $\frac{1}{2}$, 0, and $\frac{1}{2}$.

Case 3. B fires before A, and hits A with his first shot. C and B now exchange shots until one or other is hit. C survives if his first shot hits B (probability $\frac{1}{2}$), or if his second shot hits B (probability $\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{2}$; the first factor is the probability that C's first shot missed, and the second factor is the probability that B's answering shot missed, whilst the last factor is the probability that C's second shot found the mark), or in general if his nth shot hits B (probability $\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{5} \dots \frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{10^{n-1}}$).

His total probability of escape is therefore

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{10^2} + \frac{1}{2} \cdot \frac{1}{10^3} + \dots = \frac{1}{2}(1+.1+.01+.)$$

$$= \frac{1}{2} 1 . \dot{1}$$

$$= \frac{5}{9}$$

Hence the survival chances if case 3 applies are respectively

$0, \frac{4}{9}$, and $\frac{5}{9}$ for A, B and C.

The probabilities that Case 1, Case 2 or Case 3 applies are respectively,

$$\frac{1}{2}, \; \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}, \text{ and } \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}.$$

Hence the overall survival chances are

$$A - \frac{1}{2} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2} + \frac{2}{5} \times 0 = 30\%$$

$$B - \frac{1}{2} \times 0 + \frac{1}{10} \times 0 + \frac{2}{5} \times \frac{4}{9} = 17.\dot{7}\%$$

$$C - \frac{1}{2} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2} + \frac{2}{5} \times \frac{5}{9} = 52.\dot{2}\%$$

Almost Correct. J. Tong (Waverley Coll),
The following contributions missed the point that C would deliberately miss, but were otherwise correct.
F. Sharpe (Epping B.H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), W. Bundschuh (West Wallsend H.S.), P. O'Sullivan (Randwick B.H.S.), P. Jablon (Telopea Pk. H.S.).

087 Three prisoners A, B and C occupy condemned cells in a certain gaol. A questions the warden concerning a rumour that one of them is to be pardoned. The warden confirms the rumour but says that he has not been given authority to announce which prisoner is to live. A, finding the warden adamant, at last prevails upon the warden to name one of the other two prisoners who is going to be executed. "If they are both going to die", he says, "say whichever of B and C you please". The warden sees nothing wrong with the suggestion and tells A that B has not been pardoned. A now claims that his probability of survival has increased from $\frac{1}{3}$ to $\frac{1}{2}$. He manages to slide a note with the information under the wall into the next cell, occupied by C, who also congratulates himself that his survival chances have now

increased from $\frac{1}{3}$ to $\frac{1}{2}$.

Is A's reasoning correct? If not, what are the survival chances of each of the three prisoners?

Answer. A's survival chances have remained unchanged at $\frac{1}{3}$. C's chances, however, are now $\frac{2}{3}$.

Before the warden speaks, the following are possibilities:-

A pardoned - warden names B (probability $\frac{1}{6}$)

- warden names C (" $\frac{1}{6}$)

B pardoned - warden names C (" $\frac{1}{3}$)

C pardoned - warden names B (" $\frac{1}{3}$)

Since the warden in fact names B, only the first and last possibilities remain but of these the latter is $\frac{1}{3}/\frac{1}{6} = 2$ times as likely as the first.

Correct answers from: J. Tong, (Waverley Coll), F. Sharpe (Epping B.H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), P. O'Sullivan (Randwick B.H.S.).

088 Problem 8 Work out the simplest continued fraction, that of $\frac{\sqrt{5}+1}{2} = \alpha$.

Problem 9 Show that if α is rational then it has only finitely many very economical approximations p/q, i.e. so that

$$|\alpha - p/q| < \frac{1}{q^2}$$

Answer. Problem 8 $1 < \alpha < 2$ so that $a_o = [\alpha] = 1$ and $t_o = \alpha - a_o = \frac{\sqrt{5} - 1}{2}$.

Therefore $\alpha_1 = \frac{1}{t_o} = \frac{2}{(\sqrt{5}-1)} \frac{(\sqrt{5}+1)}{(\sqrt{5}+1)} = \frac{\sqrt{5}+1}{2}$.

Since this is equal to α, we see at once that $a_1 = [\alpha_1] = [\alpha] = a_o = 1$, $t_1 = \alpha_1 - a_1 = \alpha - a_o = t_o$ and in fact that the whole process repeats, yielding $\alpha_n = \alpha$, $a_n = 1$ for every n. The answer is therefore $\alpha = [1,1,1,\ldots] = [\dot{1}]$

Problem 9 If $\alpha = h/k$ where $k > 0$ and h k are relatively prime integers, one (exact) "best approximation" is α itself. We look for other approximations, p/q, where $pk \neq qh$ (p, q integers, $q > 0$).

If $q \geq k$, $|h/k - p/q| = |(hq - pk)|/kq \geq$

$$1/kq \geq 1/q^2 .$$

Hence there are no best approximations other than α itself with $q \geq k$. If $1 \leq q < k$, there exists an integer p_1 such that $p_1/q < \alpha < (p_1+1)/q$.

For all integers p other than p_1 and (p_1+1)

$$|\alpha - \frac{p}{q}| > 1/q \geq 1/q^2$$

Hence there are at most two "best approximations with any given denominator $q < k$. (There may well be fewer than two).

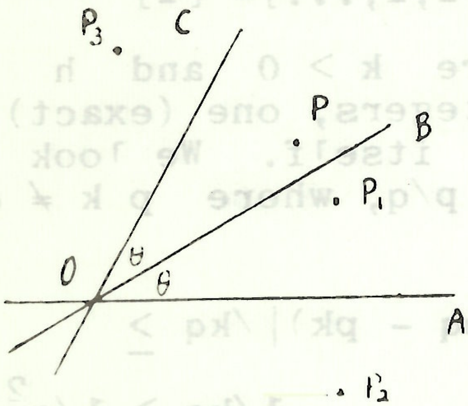
Hence there are only a finite number of best approximations to α; indeed, at most $2(k-1)$ with denominator less than k, and

α itself, making a possible maximum of $(2k-1)$ best approximations.

Correct answers received from: (i) J. Tong (Waverley Coll.) H. Clarke (Forest H.S.), D. Nash (Parkes H.S.), (i) and (ii) C. Cosgrove (Kingsgrove Nth. H.S.), P. Jablon (Telopea Pk. H.S.), P. O'Sullivan (Randwick B.H.S.).

89 Prove that if a plane figure has exactly two axes of symmetry, they are at right angles.

nswer. Suppose OA and OB are two axes of symmetry inclined at an angle θ. Let $\angle BOC = \theta$ (see fig.) We shall complete the proof by showing that OC is another axis of symmetry (which clearly coincides with OA only if θ is a right angle). Let P be any point on the figure. Then P_1, the mirror image of P in OB, is on the figure since OB is an axis of symmetry.



Likewise P_2 (the mirror image of P_1 in OA), and P_3 (the mirror image of P_2 in OB) lie on the figure. It is an easy matter to show that P_3 is the mirror image of P in OC (for example, show that $OP = OP_1 = OP_2 = OP_3$ and

$\angle COP_3 = \angle BOP_3 - \angle BOC = \angle BOP_2 - \angle BOA = \angle AOP_2$
$= \angle AOP_1 = \angle AOB - \angle BOP_1 = \angle BOC - \angle BOP = \angle COP$).

Since the mirror image in OC of an arbitrary point P on the figure lies on the figure OC is an axis of symmetry.

Correct answers received from: J. Tong (Waverley Coll H. Clarke (Forest H.S.), C. Cosgrove (Kingsgrove Nth. H.S.), P. Jablon (Telopea Pk. H.S.), D. Nash (Parkes H.S.), D. Hermann (Oak Flats H.S.), P. O'Sullivan (Randwick B.H.S.).

090 A tropical island is inhabited by 5 men and a pet monkey. The men collect a large pile of coconuts, intending to distribute them equally amongst themselves on the following morning. During the night, however, one man, A, quietly goes to the coconuts and divides them into 5 equal piles. There is one coconut left over which he gives to the monkey. He then puts four of the piles back together and hides the fifth pile before going back to bed. Later a second man, B, does exactly the same thing and so do C, D and E; i.e. each divides the heap left by the previous visitor into 5 equal piles, finds one coconut left over which he gives to the monkey, hides one pile and heaps together the remaining four. In the morning, the residual heap of coconuts is divided equally amongst the five men, one coconut left over being again given to the monkey.

What is the smallest number of coconuts which could have been contained in the original heap?

Answer Suppose that there were n coconuts in the original heap. Then from the way that A divides the coconuts,

$$n = 5n_1 + 1 \text{ , where } n_1 \text{ is an integer,}$$

and similarly

$$4n_1 = 5n_2 + 1$$
$$4n_2 = 5n_3 + 1$$
$$4n_3 = 5n_4 + 1$$
$$4n_4 = 5n_5 + 1$$

$4n_5 = 5n_6 + 1$, where $n_2, n_3 \ldots n_6$ are integers.

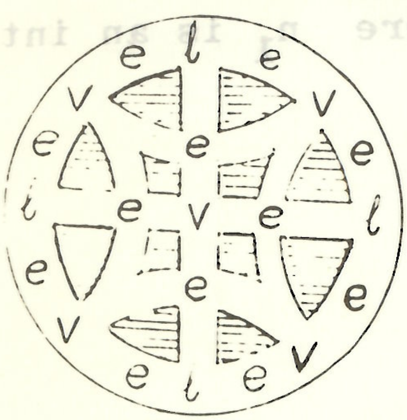
From these equations, $n = 4\left(\frac{5}{4}\right)^6 n_6 + \left(\frac{5}{4}\right)^5 + \left(\frac{5}{4}\right)^4 + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^2 + \frac{5}{4} + 1 = 4\left(\frac{5}{4}\right)^6 (n_6+1) - 4$.

Since n must be an integer, the least possible value of n_6 , giving the least possible value of n, is 4^5-1 , giving n as $5^6-4 = 15621$.

Therefore the smallest number of coconuts which could have been contained in the original heap is 15,621. (P. O'Sullivan).

Correct answers received from: J. Tong (Waverley Coll.), F. Sharpe (Epping B.H.S.) C. Cosgrove (Kingsgrove Nth. H.S.), W. Bundschuh (West Wallsend H.S.), H. Horiuchi (Vaucluse B.H.S.), D. Hermann (Oak Flats H.S.), P. O'Sullivan (Randwick B.H.S.).

Level Crossings



In how many ways can the word level be read, by moving along straight paths, curved paths or paths with corners, if the initial and final letters (ℓ) are not to coincide?

(Answer p. 32)

Sins of Commission and Omission

Commission Mr. J.A.B. Forster, of Emerald, Victoria, has drawn our attention to an error in the solution of problem 068 which appeared in Parabola Vol.3, No. 3. The reason, given in the first paragraph of the proof, for one vertex, R, of the triangle to lie on the side AD of the square, is simply not correct. Mr. Forster suggests that the reason given ought to be replaced by one which takes account of the lengths of the sides of the two figures, as well as of the angles, and we leave it to interested students to do this.

Omission In the solution of problem 080 (Parabola Vol.4, No. 1) it was tacitly assumed that it was impossible to be sure of ordering the five unequal weigh-ts in fewer than 7 weighings. This is true, but one of our contributors, P. O'Sullivan, found an argument to prove it explicitly, essentially as follows. There are $5! = 120$ possible wasy in which the five weights are ordered. At each weighing one of two results is possible. i.e. the L.H. weight is the heavier, or it is the lighter. Therefore in six weighings at most $2^6 = 64$ results are possible. Since this is less than 120, more than one ordering of the weights must yield the same results of the six weighings for at least one such set of results. This argument was generalised by O'Sullivan to show that $2^{W_n} > n!$ for any number of unequalweights n. He combined this with the inequality for W_n set as a problem in 080, to show that W_n is of the same order of magnitude as $n\log n/\log 2$; in fact, that the ratio of this number to W_n approaches the limit 1 as n becomes large.

Lost Digits (Answers p. 32)

Restore the obliterated digits in the following calculations

```
(i)   √xxxxxx = xxx   A square root
     -  x
        xxx
     -   xx
         xoxx
     -   xxxx
         ....
         ====
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                   x8x
(ii)      xxx)xxxxxx      A long division
              xxxx
                xxx
                xxx
                xxxx
                xxxx
                ....
                ====
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Mathematical Precision

A biologist, a physicist and a mathematician chanced to observe from their train compartment shortly after crossing the border into Victoria, a paddock containing a black sheep.

Hats

Five men, A,B,C,D and E, are standing in that order in a queue, E being at the head of the queue, and A bringing up the rear. From a bag which contains, as they all know, 4 white hats and 5 black hats, a sixth person places a hat on the head of each of the five.

A looks at the four people ahead of him, and announces that he cannot tell the colour of the hat that has been placed on his head. The next in line, B, considers for a moment, looks at C,D and E, and makes a similar announcement. Then C who can see only D and E says the same thing, and D after reflection admits that he also is unable to tell the colour of the hat given to him. (Of course, he can see E's hat).

E, who knows that the others are all gifted logicians, now confidently announces the colour of his hat. What colour does he announce, and what is his reasoning?

(Answer p. 32).

"Oh, look!" said the biologist, "Victorian sheep are black".

The physicist was horrified at the scantiness of the evidence available to his friend for arriving at this conclusion.

"At least one Victorian sheep is black" he corrected primly.

But even this did not satisfy the mathematician. His contribution:- "At least one Victorian sheep is black - on at least one side".

The Mathematics Competition

This annual event attracted over 1,100 entrants this year, the largest enrolment so far. The senior division of the competition was tackled very efficiently by several students, two of whom answered all questions completely. The winner, C.M. Cosgrove, of Kingsgrove North High School, included generalisations of two questions which gained for him a very narrow victory over the second prize winner, R.W. Holgate, of Sydney Grammar School, (who incidentally was a joint winner of last years senior division).

Third prize was shared by H.C. Craft (St. Andrews Cathedral School), P.E.A. Koppstein (Sydney Church of England Grammar School), and J.E. Dawson, (Sydney Grammar School), the latter being the winner of last years junior division. Congratulations to all these gifted mathematicians, and to the other prize and certificate winners, amongst whom it was pleasing to see the names of several contributors to Parabola.

The Junior division examination, answers to which appear in this issue of Parabola, proved less satisfactory. The joint winners, J.B. Wakeley, (Epping B.H. S.) and I.A. Pollard, (S.C.E.G.S. North Sydney), each succeeded with three questions and S. Williams (Manly Girls H.S.) answered two questions correctly to gain third prize, but the paper was much too hard for most competitors. Indeed, question five was the only question to be handled satisfactorily by more than a handful of students, and the large majority of prize or certificate winners solved it correctly with or without displaying some insight into other questions on the paper.