Solutions of Senior Competition Problems

Qu. 1. If a, b, c, are any three numbers show that

$$a^2 + b^2 + c^2 \geq ab + bc + ca .$$

Answer. $2\,[a^2 + b^2 + c^2 - ab - bc - ca]$

$$= (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0 .$$

(Hence equality only applies if $a = b = c$).

Qu. 2. P is a given point lying within the arms of an acute angle AOB . Show how to construct the straight line through P which cuts off the triangle of minimum area.

Answer. Construct $PX \parallel OB$ cutting OA at X . Make $^*XY = {}^*OX$. Join YP and produce to cut OB at Z . YZ is the desired line. To prove this let any other line Y_1Z_1 through P (if necessary produced) cut the line through Y parallel to OB at U . Then $\Delta YPU \equiv \Delta ZPZ_1$ whence it is obvious that area of $\Delta OYZ <$ area ΔOY_1Z_1 .

Qu. 3. Let m be an integer greater than 1 and define the numbers $m_1, m_2, m_3, \ldots$, by putting $m = m_1$ and

$$m_{i+1} = m_i^2 - m_i + 1$$

for $i = 1,2,3,\ldots.$ Show that none of the numbers $m_2, m_3, m_4, \ldots$ is divisible by m.

Answer. m_2 is expressible in the form $m_2 = m\,q_2+1$ where q_2 is an integer. In fact $q_2 = (m-1)$. Also if for any k, $m_k = m\,q_k + 1$ where q_k is an integer, then

$$m_{k+1} = m_k^2 - m_k + 1$$
$$= m[m\,q_k^2 + q_k] + 1$$
$$= m\,q_{k+1} + 1$$

where $q_{k+1} = m\,q_k^2 + q_k$ is an integer.

It follows by mathematical induction that for every $i > 1$ $m_i = m\,q_i + 1$ whence m_i is not divisible by m.

Qu. 4. Five players each hold five cards of values 1,2,3,4 and 5 respectively. Each player exchanges one of his cards with each of the other four players, and in each case receives a card of the same value in return. Show that among the five cards which are not exchanged there is one of each value.

Answer. Of the five cards valued 1, an even number are exchanged (since at each exchange exactly 2 change hands). Therefore an odd number remain not exchanged. Hence at least one card valued 1 is not exchanged. Similarly there is at least one card valued 2, 3, 4 or 5 not exchanged. As the number of cards not exchanged is equal to 5 , there must be exactly one card of each value not exchanged.

Qu. 5. In a certain country there are 5 cities which are connected by a system of roads such that:

(i) each road joins two cities; the only road junctions occur at the cities;

(ii) for any pair of cities there is exactly one route which a motorist can follow from one city to the other.

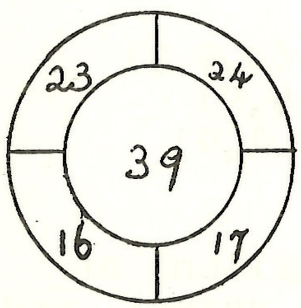
Prove that there are exactly four roads joining the cities.

If there are n cities, prove that there are exactly $n - 1$ roads.

Answer. We shall prove the result for n cities by induction. If $n = 2$, there must clearly be only 1 road.

Suppose for k cities such a system of roads necessarily consists of $(k-1)$ roads. We shall prove that it then follows that for $(k+1)$ cities such a system of roads consists of exactly k roads. Let A_1 be any one of the $(k+1)$ cities. If there is only one road leaving A_1 , we delete both A , and this road. The remaining k cities are connected by the remaining roads which still must satisfy conditions (i) and (ii) . (For (ii) the unique route joining any two of the remaining towns cannot have required use of the deleted road since there was no other exit from A_1). By the induction hypothesis (k-1) roads remain, so there were k originally. If there are several roads leaving A_1 we travel along any of them to A_2, and then if possible by a different road to A_3 (which cannot be the same as A_1). Eventually since there are only n towns, we must reach a town P which has no other road leaving it than the one by which it was reached, and this town can be used in place of A in the above induction argument to complete the proof.

Darts (Answer p.40)



On this dartboard I scored exactly 100. How?

Can You Do It Duck? (Answer P.40)

Many years ago, long before decimal currency, a farmer went to market and bought 100 animals for £100. His purchase consisted of cows at £5 each, sheep at £1 each and ducks at 1/- each. How many of each did he buy?

Answers to the Puzzles.

Puzzle (p 10)

A = 9, B = 7, C = 1, D = 3, E = 4, F = 8, G = 2, H = 0.

Darts (p 39)

I scored 17 four times and 16 twice.

Can Ewe Do It Duck? (p 39)

The farmer bought 19 cows, 80 ducks and 1 sheep.

The Wine Vessels (p 16)

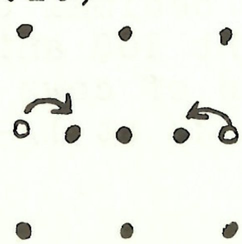
(20,0,0) → (11,0,9) → (11,9,0) → (2,9,9) → (2,13,5) → (15,0,5) → (15,5,0) → (6,5,9) → (6,13,1) → (19,0,1) → (19,1,0) → (10,1,9) → (10,10,0).

The triplets give the number of gallons of wine in the 20 gal vessel, the 13 gal. vessel, and the 9 gal. vessel in that order, at successive stages of the solution.

High Finance (p 15)

36 dollars was distributed amongst 6 children.

Coins (p.20)



One answer is indicated by the accompanying diagram.