PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions and the names of those who send correct solutions by January 31st, 1968, will be published in the next issue of PARABOLA.

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior Section. All students may submit solutions of problems in the Open Section.

JUNIOR

J101 If n is a positive integer other than 2, 3 or 5, any square can be dissected into n smaller squares.

J102 When 31,513 and 34,369 are divided by a certain three digit number, the remainders are equal. Find this remainder.

OPEN

O103 The number .202532 is to six decimal places a fraction a/b where a and b are positive integers less than 100. Find a and b.

0104 In a certain community there are a number of clubs, none of which includes the entire community in its membership. For every pair of distinct clubs there are at least two people who are members of each. No two clubs have exactly the same members; if any three people are considered there is one and only one club having all three as members. If there are at least two clubs show that -

(a) for every pair of clubs there are exactly two people who are members of both;

(b) the community contains at least four people;

(c) every club has a membership of at least three.

Can you prove anything else about the clubs?

0105 A new piece has been invented for use in the game of chess. This piece, the duke, makes moves rather similar to those made by the knight. We call the square in the r-th row and the s-th column of the chess board (r,s); a duke situated in this square may move to any of the eight squares $(r\pm3, s\pm1)$ or $(r\pm1, s\pm3)$ which are within the chess board.
If the duke is in the nearest left-hand corner, i.e. in square $(1,1)$ and there are no other pieces on the board is it

possible to move it to any of the other corner squares (1,8) (8,8) or (8,1)? If so state in each case (a) the minimum number of moves; (b) whether the total number of moves to do this is restricted in any way.
Prove the results you have stated.

<u>0106</u> The function $P^{(d)}(x,y,z)$ where d is a positive integer, is the sum of all the distinct products $x^a y^b z^c$ where a,b,c are positive integers such that $a + b + c = d$. For example

$P^{(1)}(x,y,z) = x + y + z$, and

$P^{(2)}(x,y,z) = x^2 + y^2 + z^2 + yz + zx + xy$.

If $x = 1$ and $y > 0$, $z > 0$ show that $P^{(d+1)}(x,y,z) > P^{(d)}(x,y,z)$.

If $x = 1$, $y = \frac{1}{4}$, and $z = \frac{1}{9}$ show that $P^{(d)}(x,y,z) < 1.5$ for all finite d.

<u>0107</u> Sketch the curve $y = x^{-1/2}$ between $x = 0$ and $x = 1$ and calculate

$$\int_0^1 x^{-1/2}dx.$$

Using the above result, or otherwise, show that if n is a positive integer, the expression

$$\frac{\sqrt{2}}{\sqrt{n}}\left\{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} + \dots \frac{1}{\sqrt{2n-1}}\right\}$$

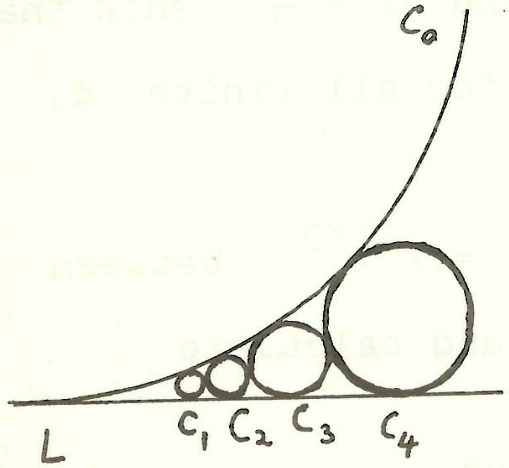
lies between 1 and 2.

0108 Prove that if the integers $a_1, a_2, \dots a_n$ are all distinct, then the polynomial

$$(x-a_1)^2(x-a_2)^2(x-a_3)^2 \ . \ . \ . \ (x-a_n)^2 + 1$$

cannot be written as the product of two other polynomials with integral coefficients.

0109



C_o is a circle of diameter D, touching a line L. The circles $C_1, C_2, C_3, \dots, C_n$ are constructed in succession each $C_i^{(i>1)}$ touching C_o, L, and C_{i-1}. (C_1 is any circle touching both L and C_o.) If the diameter of C_i is equal to D_i, find a formula for D_n in terms of D and D_1.

O110 Let N be an arbitrary natural number. Prove that there exists a multiple of N which contains only the digits 0 and 1. Moreover, if N is relatively prime to 10 (that is, is not divisible by 2 or 5), then some multiple of N consists entirely of ones. (If N is not relatively prime to 10, then, of course, there exists no number of form 11 ... 1 which is divisible by N.)

High Finance

A sum of money was divided among some children. The first child received 1 dollar and one seventh of the remainder, the second received 2 dollars and one seventh of the new remainder, the third 3 dollars and one seventh of the new remainder, and so on. When all the money was gone all the children had equal shares. How much money was distributed amongst how many children?

(Answer p.40)

The Tower of Hanoi:- (p.28)

Answer If N_r is the minimum number of moves required to shift r discs from peg A to peg B it is easy to see that $N_{r+1} = 2.N_r + 1$. For the top r discs must first be transferred to peg C which requires N_r moves, then the (r+1)th disc is placed on B, and finally, using another N_r moves, the r discs on peg C are transferred to peg B.

It is obvious that $N_1 = 1$, i.e. if there is only one disc it may be transferred to peg B on the first move. It then follows that

$$N_2 = 2.1 + 1 = 3$$

$$N_3 = 2.3 + 1 = 7$$

$$N_4 = 2.7 + 1 = 15$$

and

$$N_5 = 2.15 + 1 = 31$$

In fact, a general formula for N_n is $2^n - 1$, as may be readily proved by induction.

The Wine Vessels

Three vessels are of capacity 20 gals, 13 gals and 9 gals. The first is full of wine, the others empty. Show how to halve the wine, with 10 gals in each of the larger two vessels, if no other measuring vessel is available.

(Answer p.40)