Answers to Problems in V.4 No. 2.

J91 (i) Find all whole numbers such that when the third digit is deleted, the resulting number divides the original one.

(ii) Find all whole numbers with initial digit 6 such that when this initial digit is deleted, the resulting number is 1/25 of the original number.

Answer. (i) Let $x = a_1 a_2 a_3 \ldots a_n = a_1 10^{n-1} + a_2 10^{n-2} + a_3 10^{n-3} + \ldots + a_n$ be such a number. Then $y = a_1 10^{n-2} + a_2 10^{n-3} + a_4 10^{n-4} + \ldots + a_n$ is the number which results when the third digit (a_3) of x is deleted, and

$$x - 10y = (a_3-a_4)\, 10^{n-3} + (a_4-a_5)10^{n-4} + \ldots + a_n .$$

Since $0 \le a_k \le 9$ for $k = 1,2,\ldots,n$, we have

$$|x-10y| \le |a_3-a_4|10^{n-3} + |a_4-a_5|10^{n-4} + \ldots + (a_n)$$

$$\le 9.10^{n-3} + 9.10^{n-4} + \ldots + 9$$

$$< 10^{n-2} \le y$$

But we are given that y divides x,

and therefore also $|x-10y|$, a non negative integer smaller than y. The only possibility is $|x-10y| = 0$. Comparing the digits of $10y = a_1 10^{n-1} + a_2 10^{n-2} + a_4 10^{n-3} + \ldots + a_n 10' + 0$. with those of x, we obtain $a_3 = a_4$, $a_4 = a_5$, $a_5 = a_6, \ldots, a_{n-1} = a_n$, $a_n = 0$. Hence all digits of x from the third on must equal zero.

(ii) Let $x = 6 \times 10^n + y$ where $y < 10^n$ be such an integer. Then $x = 25y$

$$6 \times 10^n + y = 25y$$

$$24y = 6 \times 10^n$$

$$y = \frac{1}{4} \times 10^n = 25 \times 10^{n-2}$$

Hence, the smallest such number occurs when $n = 2$, giving $y = 25$, $x = 625$, and all other numbers are obtained by appending zeros. i.e. 6250, 62500 etc.

Correct answers received from: B. Cottee (Petersham G.H.S.), N. Wormald (Sydney Tech. H.S.)., B. Tweedie (Leeton H.S.), I. Fay (Wallsend H.S.).

<u>J92</u> Solve the equation

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x+2$$

[The symbol $|a|$ is called the absolute value of a and is defined by $|a| = a$ if a is positive, $|a| = -a$ if a is negative, so that $|a|$ itself is never negative].

<u>Answer</u> If $x \leq 1$, all of $(x+1)$, x, $(x-1)$ and $(x-2)$ are negative so the equation becomes

$-(x+1) - (-x) -3(x-1) --2(x-2) = x+2$

which has $x = -2$ as a solution.

If $-1 < x \leq 0$, $x+1$ is positive and the equation becomes $(x+1) -(-x) -3(x-1) --2(x-2) = x+2$ which has no solution.

If $0 < x \leq 1$, the equation becomes

$(x+1) - x - 3(x-1) +2(x-2) = x+2$

which has no solution in the relevant range. (The solution $x = -1$ is not between 0 and 1).

If $1 < x \leq 2$ the equation is

$(x+1) - x + 3(x-1) + 2(x-2) = x+2$ which has the solution $x = 2$.

If $2 \leq x$ the equation is

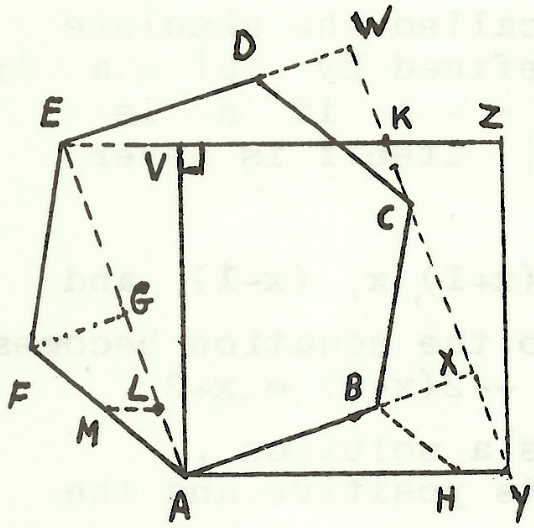
$(x+1) - x + 3(x-1) -2(x-2) = x+2$ i.e. $0x = 0$

which is satisfied by all values of x (in the range $x \geq 2$). Hence the solutions of the equation are -2 and all numbers ≥ 2.

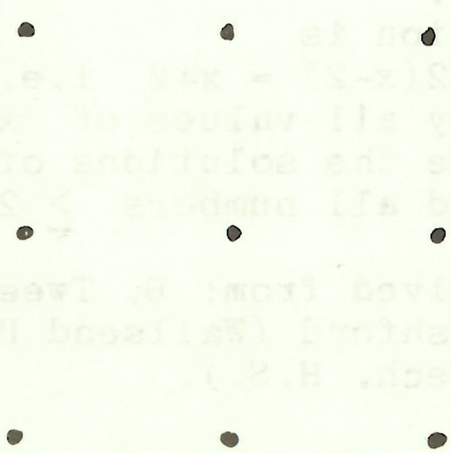
Correct answers received from: B. Tweedie, (Leeton H.S.), S. Bashford (Wallsend H.S.)., N. Wormald (Sydney Tech. H.S.).

093 Show how to dissect a regular hexagon into 5 pieces which can be rearranged to form a square.

Answer. Let ABCDEF be a regular hexagon of side 1 unit and area $\frac{\sqrt{27}}{2}$ units. Join EA, bisect it at G and join FG. If the FGA is moved to position DWC and EFG to position CBX, the hexagon is transformed into the rectangle AXWE. This may be transformed into a square by constructing K on WC with EK = $\frac{\sqrt[4]{27}}{\sqrt{2}}$ units. Then construct AY parallel to EK, and AV and YZ perpendicular to EK produced. This gives a dissection of the hexagon into six pieces which reassemble to form the square. However, two of these, CBX (EFG), and BXYH (FGLM) can be joined up again in both figures (delete the cuts FG and BX) to obtain the desired result.



Coins



In this diagram 9 coins are arranged in eight rows each containing 3 coins (notice the two diagonals). Move two coins to new positions so that there are now 10 straight lines each passing through the centre of three coins.

(Answer p.40)

<u>094</u> Show that every triangle can be dissected into 7 acute angled triangles.

<u>Answer</u>. If the triangle is itself acute angled there is no difficulty since for example joining the mid points of the sides produces a dissection into 4 triangles each similar to the original. Repeating this operation on one of the smaller triangles yields a dissection into seven acute angled triangles

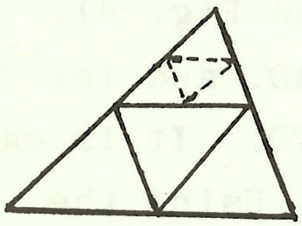


Fig 1

If the triangle is right angled or obtuse angled it is not possible to dissect it into fewer than seven triangles which are all acute angled. For, the obtuse angle must be divided by (at least one) cut AP. If this were to reach the opposite side BC a further obtuse angled triangle would result (or two right angled triangles) and we would be no nearer a dissection into acute angled triangles than when we started. Hence the cut AP must terminate at a point P in the interior of ΔABC . The 360^o angle at P must be divided into at least 5 angles by further cuts. If two of these extra cuts emanating from P were to intersect AB, a further obtuse angled (or right angled) triangle would certainly result. Hence one only intersects AB, and similarly one only intersects AC, leaving two to intersect BC.

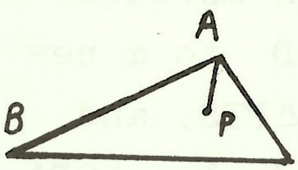


Fig 2

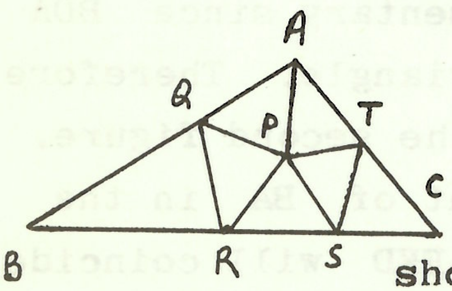


Fig 3

Completing cuts QR and ST in Fig. 3 now produces a dissection into seven triangles which appear to have a chance of being all acute angled if P,Q,R,S and T are suitably chosen. We proceed to show that such a choice is always possible. We use the intuitively obvious fact that if one vertex of a triangle is moved continuousl; along a path the angles of the triangle vary continuously (i.e. small movements of one vertex result in small changes of the angles of the triangle.)

Let ABC be obtuse angled (or right angled) at A and let $\angle B \leq \angle C$ (See Fig. 4). Construct AD $\perp$ BC, DQ $\perp$ AB, QF $\perp$ AD. and let QF produced cut AC at T. Join TD. It is easy to see that $\angle CTD \leq 90^o$. (Prove). Using the above mentioned fact it is clear that points R on BD and S on DC can be chosen so close to D that ΔBQR, ΔTSC, and ΔQFR ΔFTS and ΔFRS are all acute angled. If now F is moved a sufficiently small distance down AD to a new position P, the triangles ΔQPR, ΔPTS, and ΔPRS will still be acute angled and also ΔQPA and ΔTPA will now be acute angled.

095 Prove that the four pieces of Dudeney's dissection of the equilateral triangle do fit together exactly to make a square.

Answer. $\angle$KDB and $\angle$ADM are supplementary since BDA is a straight line in the triangle. Therefore KDK is a straight line in the second figure. Also as D was the mid-point of BA in the triangle the vertex A of AEKD will coincide with B of BDKA in the second figure.

(cont.p.24)

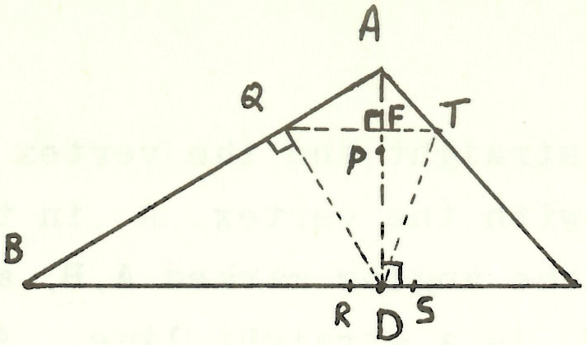


Fig 4
(Answer 094)

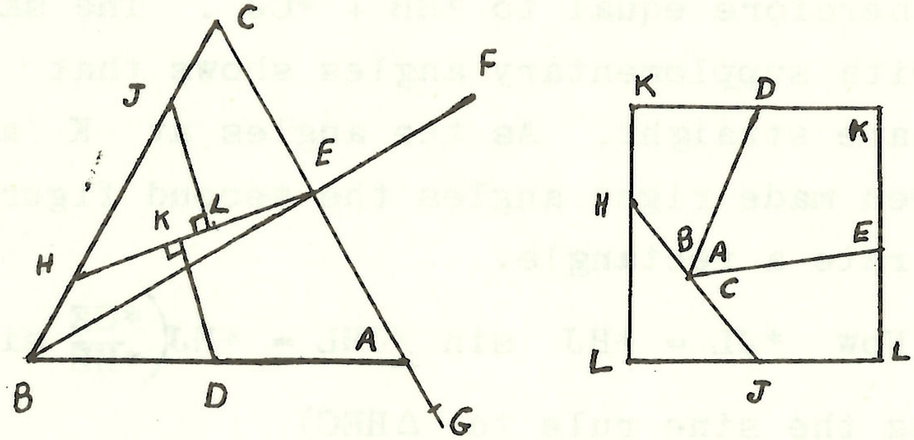


Figure for Answer 095

Given ΔABC is equilateral, $^*AE = {}^*EC = {}^*BD = {}^*DA = {}^*HJ = {}^*EF$. The semicircle on diameter BF intersects CA produced at G ; $^*EH = {}^*EG$; $DK \perp HE$; $JL \perp HE$.

Prove That this dissection of the ΔABC reassembles to form a square as suggested by the figure.

Similarly, KEL is straight and the vertex C of CJLE coincides with the vertex A in the second figure. As the angles marked A, B, and C are all 60° HBJ is a straight line. Also

*HJ (in ΔHLJ) was constructed equal to half *BC and therefore equal to *HB + *CJ . The same argument with supplementary angles shows that KHL and LJL are straight. As the angles at K and L have been made right angles the second figure is at any rate a rectangle.

$$\text{Now} \quad {}^*JL = {}^*HJ \ \sin \angle JHL = {}^*HJ\left(\frac{{}^*CE}{{}^*HE} \sin 60^\circ\right)$$

(by applying the sine rule to ΔHEC)

$$\therefore {}^*JL = \frac{\frac{1}{2}\,a.\ \frac{1}{2}\,a.\ \ \frac{\sqrt{3}}{2}}{\sqrt{\frac{\sqrt{3}}{2}\,a.\frac{1}{2}\,a}}$$

where a is the length of the side of the triangle. (We have used the fact that

$$({}^*HE)^2 = ({}^*EG)^2 = {}^*BE\ {}^*EF = \frac{\sqrt{3}}{2}\,a.\frac{1}{2}\,a\ .)$$

$$\text{i.e. } 2{}^*JL = \frac{a}{2}\ \frac{\sqrt{3}}{{}^4\!\sqrt{3}} = \frac{{}^3\!\sqrt{4}\ a}{2}$$

$$\therefore\ {}^*KH + {}^*HL \text{ (in the rectangle)} = \frac{\text{Area of } \Delta}{2{}^*JL}$$

$$= \frac{\frac{1}{4}\sqrt{3}\,a^2}{\frac{4\sqrt{3}\,a}{2}} = \frac{4\sqrt{3}\,a}{2}$$

Hence the rectangle is a square.

<u>096</u> Show that $\sum_{k=1}^{n} \frac{1}{k}$, i.e. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$

is never exactly equal to a whole number for any integer n greater than 1.

<u>Answer</u>. Let ℓ be the highest integer such that $2^\ell \leq n < 2^{\ell+1}$. The lowest common multiple of $1,2,3,\ldots n$ is equal to $2^\ell D$ where D is odd.

Then $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^\ell} + \ldots + \frac{1}{n} =$

$$\frac{2^\ell D + 2^{\ell-1} D + \ldots + D + \ldots}{2^\ell D} \quad (1)$$

Of the terms on the numerator all are even except the one resulting from the fraction $\frac{1}{2^\ell}\left(= \frac{D}{2^\ell D}\right)$ since other numbers $k \leq n$ are not divisible by 2^ℓ in view of the choice of ℓ

Thus $k = 2^r m$ where m is odd $r < \ell$ and

$$\frac{1}{k} = \frac{2^{\ell - r} \frac{D}{m}}{2^{\ell} D} \quad \text{where } \frac{D}{m} \text{ is an integer.}$$

The numerator on the R.H.S. of (1) is therefore an odd number and therefore it cannot be an integral multiple of the even denominator.

Correct answers received from: P. Jablon (Telopea Pk. H.S.).

097 A game is played with two people and a heap of matches. There are initially an odd number of matches in the heap. The players move alternate-ly, each move consisting in the removal of 1, 2, or 3 matches from the heap. The winner is the player who ends up with an odd number of matches when there are no matches remaining in the heap. If there are initially 61 matches in the heap which player ought to win with correct play, the first to move or the second? Describe his winning strategy.

Answer. The strategy of the game is contained in the following table, which can be extended further as desired.

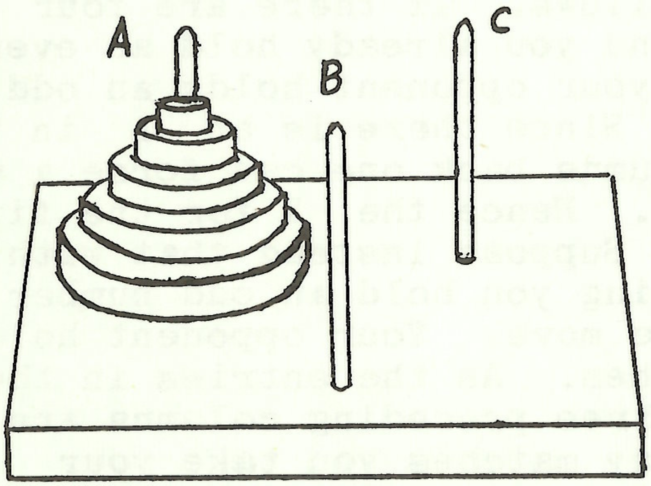
	1	2	3	4	5	6	7	8	9	10	11	12
Even	W	W	W	W	L	W	W	L	W	W	W	W
Odd	L	W	W	L	W	W	W	W	L	W	W	L

By way of example, suppose you have previously taken an even number of matches, there are 9 matches left, and it is your turn to play. The entry in the table in the row marked even in column numbered 9 is a W and this means that with correct play you should win. Your correct play consists in taking 1 match. Then your opponent holds an even number of matches (why?) with eight remaining and it is his turn to play, so the L in the relevant place in the table shows that his game is lost whatever play he makes if you continue to play correctly.

The table may be constructed column by column from the left hand end. The first three columns are easy. One then constructs the fourth column as follows. If there are four matches remaining and you already hold an even number of matches, your opponent holds an odd number of matches. Since there is an L in the "odd" row three columns back one can force a win by taking 3 matches. Hence the W for the first entry in column 4. Suppose instead that with four matches remaining you hold an odd number of matches and have the move. Your opponent holds an even number of matches. As the entries in the "even" row in the three preceding columns are all W, however many matches you take your opponent is left with a won game. Hence the L appearing as the lower entry in column 4.

Notice now that the columns 9, 10, and 11 have the same entries as columns 1, 2, and 3 [and corresponding columns are headed by numbers of the same parity]. Since it is only the entries in these three columns which are used to construct the entries in column 12, this must be identical with column 4. In fact it is clear that the first eight columns must be repeated indefinite-ly to extend the table. In particular the 61st column will be identical with the fifth (since 61 = 8.7 + 5) . Since the first player has as yet no matches, and nought is even, the L in the fifth column shows that he cannot win if the second player plays correctly, viz. by following the strategy deducible from the table.

The Tower of Hanoi.



This is not a North Vietnamese observation post, as you might suppose, but is the name of the following mathematical game or puzzle. In the diagram there are three spikes A, B and C, the former of which holds a "tower" of discs of different sizes as illustrated. It is required to shift this tower to spike B by a succession of moves each of which consists in transferring one disc from one spike to another. One condition which must be observed, however, is that no disc may be placed on a spike which already holds a smaller disc. How many moves are required if there are initially 5 discs on spike A? How many moves if there are n discs on spike A to start with?

(Answer p. 16)

098 Another game is played with two people as follows. The first player, A, writes down on separate pieces of paper any number of distinct integers. He then turns these face down on a table. The second player, B, does not know which numbers have been written down. He begins turning the pieces of paper over one at a time. At any stage he can stop and say "This number which I have just turned face up, is the largest number you wrote down". If he is right he wins the game, otherwise he loses. What is B's best strategy, if A has used 20 pieces of paper, or n pieces of paper? What is his probability of winning?

Answer. B's only possible strategy consists in looking at k of the numbers without comment and noting the largest one M_k. Thereafter B does not speak if a number less than M_k appears, since it cannot be the largest. However, if a number greater than M_k occurs, he immediately declares it to be the largest.

The question is, "What is the best choice of k?" if there are n pieces of paper. B's probability of success for a given choice of k can be calculated as follows. His strategy wins if the largest number of the set is the $(k+r)$th which he turns over, always providing that he has not turned another number greater than M_k before he reaches the right one.

The probability that the (k+r)th number turned is the largest written down is $\frac{1}{n}$ for each $r = 1,2, \ldots, n-k$. The probability that the largest number in the first (k+r-1) turned is actually among the first k is $\frac{k}{k+r-1}$

Hence his total probability of success is

$$P_k = \frac{1}{n}\left(\frac{k}{k} + \frac{k}{k+1} + \cdots + \frac{k}{n-1}\right) = \frac{k}{n}\sum_{r=1}^{n-k} \frac{1}{k+r-1} \qquad (1)$$

It is required to choose k so that P_k is a maximum.

Note that $P_{k+1} - P_k = \frac{1}{n}\left(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{n-1} - 1\right)$

so that $P_{k+1} \gtrless P_k$ if $\frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{k+1} \gtrless 1 \quad (2)$

Let $n = 20$. Since $\frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \ldots \frac{1}{8}$

$= .955 (<1)$

and $\frac{1}{19} + \ldots + \frac{1}{7} = 1.098 > 1$ we have $P_8 < P_7$ and $P_7 > P_6$. Hence the best strategy is to select $k = 7$ and if he does this B's chance of winning is $P_7 = \frac{7}{20}\left(\frac{1}{19} + \frac{1}{18} + \ldots + \frac{1}{7}\right) = .384$

$= 38.4\%$

In fact for any n the best choice of k is that $\sum_{r=k}^{n-1} \frac{1}{r} > 1$ whilst $\sum_{r=k+1}^{n-1} \frac{1}{r} < 1$ and then (1) enables the exact probability of success to be calculated. The sólution can be pushed a little further with the use of a well-known result of analysis. viz. for large n,

$$\sum_{r=k}^{n-1} \frac{1}{r} \doteq \log_e \frac{n-1}{k-1}$$

where e, the base of natural logarithms, is approximately 2.718. For the best strategy this has to be approximately equal to 1 so that $\frac{n-1}{k-1} \doteq e$. Substituting in (1).

$$P_k = \frac{k}{n} \cdot 1 \doteq \frac{k-1}{n-1}$$ (since we are assuming n and hence k fairly large)

$$\doteq \frac{1}{e}$$

$$\doteq 36.8\%$$

This calculation shows that B's chance of winning is almost independent of the number of integers written down (provided it is not very small) if he uses the best strategy.

Almost correct: P. Jablon, (Telopea Pk.H.S.).

099 A number of towns P_1, P_2, ... P_n , no three collinear, are connected by a system of roads with the following properties. Given any pair of towns there is a straight road joining the two towns. There are no road junctions except at the towns. (Roads may cross each other at other points, but fly overs are used which make it impossible to leave one road and enter the other). All the roads are one way but the directions of traffic in the roads are so chosen that it is possible to reach any town from any other. Prove that it is possible to leave P_1 and drive in a circuit so that on returning to P_1 , every other town has been visited exactly once.

Answer. Suppose that the largest circuit starting (and ending) at P_1 visits k towns which are (relabelling if necessary) in order of visiting, $P_1, P_2, P_3, \ldots, P_k, P_1$. We effect a proof by supposing $k < n$ and producing a contradiction. If $k < n$, there is some town, A, not in the circuit. There are direct roads connecting each P_i $(i = 1,2,\ldots,k)$ with A . Suppose traffic flows from P_1 to A in the road P_1 A . We look at the direction of traffic flow in each of the roads P_2 A, P_3 A ... in turn. Either it is towards A for every i $(= 2,\ldots,k)$ or there is a number $\ell < k$ such that the traffic flows from P_i to A in the roads P_i A for $i \leq \ell$ and from A to P_ℓ in the road P_ℓ A . In this latter case we have a circuit $P_1, P_2, \ldots, P_\ell, A_1 P_{\ell+1} - P_k \ldots P_1$

containing (k+1) towns, contradicting our choice of k. Similarly if the traffic in $P_1 A$ is away from A we easily arrive at a contradiction unless this is also true for each of the roads $P_i A$.

Let us suppose then that in each of the roads $P_i A$ the traffic is towards A. It is nevertheless possible to travel from A to towns in the circuit (obviously visiting first other towns B,C.... not in the circuit). Let P_ℓ be the (or a) town in the circuit which can be reached from A by traversing a minimum number of intervening roads AB, BC,... . Then $P_1, P_2, \ldots, P_{\ell-1}$, A,B,... $P_\ell, \ldots P_k, P_1$ is a circuit with more than k towns, contradciting our choice of k. A similar contradiction is easily produced if in each of the roads P_2 A the traffic flow is away from A.

Correct answer received from: P. Jablon (Telopea Pk. H.S.), D. Nash (Parkes H.S.)

0100 Three feet four inches above the centre of an end wall of a rectangular room there is a smear of honey. (I don't know how it got there). A very mathematically minded ant emerges from a hole in the opposite end wall 3'4" below the centre. He calculates that to reach the honey there are eight different paths all 27'1" in length, and that every other route is longer.

What are the dimensions of the room?

(The idea behind this puzzle goes back again to H.E. Dudeney, but blame for an extra twist belongs to M. Kraitchik).

Answer Imagine that the room has been cut along several corners and the walls spread out in one plane. The diagram indicates several different ways in which this can be done. (Three possible positions of each end wall are indicated; one side wall has been omitted). The smear of honey is represented by Y(or Y', or Y") and the ant hole by X (or X', or X"). Possible shortest routes for the ant are given by straight lines joining an X to a Y. Note that XX' is equal in length and parallel to Y'Y" so that XX'Y"Y' is a parallellogram. Hence *XY' = *X'Y". It is easy to see that *X"Y > *XY" and that *X'Y = *X"Y' > *XY' (use Pythagoras Theorem). We are left with the following ten reasonable routes for the ant: (a) XY and X"Y" which are equal in length. (b) XY', X'Y" and two similar routes acorss the other side wall, all equal in length. (c) X'Y' and a route of equal length across the other side wall. (d) XY" and a route of equal length across the other side wall.

(For Diagram see next page).

From data, eight of these areequal in length, the other two longer. It is not possible for *X'Y' to equal *XY" since the parallellogram XX'Y"Y' would then be a rectangle so that then *XY' < *X'Y'. This would leave us with at most the 6 routes of types (a) and (b) as the shortest.

Hence it isclear that either the two routes in (c) or the two routes in (d) must be equal to the 6 routes of types (a) and (b), the other two being longer. This gives 2 possible solutions of the problem, case 1 and case 2 below.

<u>Case 1.</u> $*XY = *XY' = *X'Y' = 27'1'' = 325''$

(i) $325 = H + L \; (=*XY)$

(ii) $325^2 = (L + \frac{B}{2} + \frac{H}{2} - 40)^2 + (\frac{B}{2} + \frac{H}{2} + 40)^2$

($=*XY'^2$, using Pythagoras Theorem)

(iii) $325^2 = (L + B)^2 + 80^2$ ($*X'Y'^2$, using Pythagoras Theorem)

Solving we obtain $L = 195''$, $H = 130''$, $B = 120''$ for dimensions of the room.

<u>Case 2.</u> $*XY = *XY' = *XY'' = 325''$

Equation (iii) above must be replaced by

(iv) $325^2 = (L + H - 80)^2 + (B + H)^2$

Solving (i), (ii) and (iv) we obtain

$L = 223.20''$, $H = 101.80''$, $B = 111.75''$