PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions and the names of those who send correct solutions by July 31st, 1968, will be published in the next issue of PARABOLA.

Only those students who have not yet commenced their fourth year of secondary education are eligible to submit solutions of problems in the Junior Section. All students may submit solutions of problems in the Open Section.

JUNIOR

J111 The integer M consists of 100 threes (i. e. in decimal notation M = 3 3 3 . . . 3 with 100 digits, all 3) and the integer N consists of 100 sixes. What digits occur in the product M. N.

J112 In how many ways can 2^n be expressed as the sum of four squares of integers.

OPEN

0113 (a) If x, y and z are integers such that $x^2 + y^2 + z^2 = 2xyz$, show that $x = y = z = 0$.

(b) Find all integer solutions of $w^2 + x^2 + y^2 + z^2 = 2wxyz$.

0115 Find all solutions in integers of $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

0116 By adding parentheses a^{b^c} can mean one of two different things viz. either $a^{(b^c)}$ or $(a^b)^c = a^{bc}$

How many different meanings can be given to

$a^{b^{c^{d^e}}}$ by the addition of parentheses?

How many different meanings can be given to

$a^{b^{c^{\cdot^{\cdot^{\cdot^k}}}}}$ if there are n symbols, all different?

0117 At a certain reunion, each man present shook hands with an odd number of women and each woman present shook hands with an odd number of men. Prove that there were an even number of adults present.

0118 (Philip Hall's marriage problem).

The men in college X wish to get married to some of the girls in college Y. Each man x in college X picks out a set x' of girls in college Y, any of whom is acceptable to him; but he refuses to marry any girl not in the set x'. Under what conditions is it possible for all the men to marry? First, it is necessary that no x' be empty; but that is not enough - it could be that each man picks only one girl, and that they all pick the same girl. It is clearly necessary that, for every 2 men x and y, $x' \cup y'$ should contain at least 2 girls.

Again, this is not sufficient. It is clearly necessary that, for every subset A of X , the set

$$A' = \bigcup_{x \in A} x'$$

have at least as many elements as A. Prove that this condition is sufficient.

0119

On a desert island there are n men and n women. Each man lists the women in his order of preference, and each woman lists the men in her order of preference. Suppose they pair off and marry. This set of marriages will be called unstable if there are a man and a woman, not married to one another, each of whom prefers the other to his or her present spouse; for then these two are apt to run off together. Prove that it is possible to avoid such trouble - a stable set of marriages always exists. (Note that it does no harm if a man prefers another woman to his wife, as long as this other woman prefers her own husband.)

O120. Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$

It can be shown that there exist numbers e ($\approx 2 \cdot 17$) and c ($\approx \cdot 577$) such that S_n becomes arbitrarily close to $\log_e n + c$ for all sufficiently large values of n.

Using this, show that if

$$t_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \quad \ldots \quad - \frac{1}{2n}$$

$$= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$

then t_{2n} becomes arbitrarily close to $\log_e 2$ for sufficiently large values of n.

Can you find the "limiting sum" of the infinite

series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots .$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) ?$$

Dudeney's Map-Folding Puzzle.

1	8	3	4
2	7	6	5

(1)

1	8	7	4
2	3	6	5

(2)

1	8	2	7
4	5	3	6

(3)

To make an entertaining puzzle, divide a rectangular sheet of paper into eight squares, and number them on one side only, as shown in the first diagram at the left. The puzzle consists in folding the sheet along the ruled lines to form a square packet, in which the squares are in serial order from 1 to 8, with the 1 face up on top.

If you succeed with the first one, try the next, which is more difficult, and finally the thirdone, which is more difficult still.

(Answers p. 32)