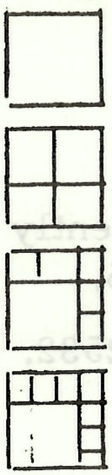
ANSWERS TO PROBLEMS IN PARABOLA VOL. 4, NO. 3

J101 If n is a positive integer other than 2, 3 or 5, any square can be dissected into n smaller squares.

Answer:



The diagrams show dissection into n squares where n = 1, 4, 6 and 8 respectively. If one of the smaller squares so obtained is dissected into 4 congruent squares the number of squares results from quartering one corner of the second diagram. Since any integer n larger than 8 exceeds 6, 7, or 8 by a multiple of 3, it is clear that a dissection into n squares may be obtained by repetition of this quartering process starting from one or other of the above diagrams.

J102 When 31,513 and 34,369 are divided by a certain three digit number, the remainders are equal. Find this remainder.

Answer:

Let the divisor involved be d, the remainder r, and the quotients q_1 and q_2 respectively.

Thus $34,369 = d.q_1 + r$

$31,513 = d.q_2 + r$

Subtracting $2,856 = d(q_1 - q_2)$.

Hence d is a three digit number which divides 2,856. There are several of these e.g. 102, 357 and many others.

However, since $34,369 = 12 \times 2,856 + 97$ it is clear

Cont.

that every three digit divisor of 2,856 will leave a remainder of 97 when it is divided into 34,369. Hence the required answer is 97.

<u>0103</u> The number .202532 is to six decimal places a fraction a/b where a and b are positive integers less than 100. Find a and b.

<u>Answer:</u>

The clue to finding the correct fraction efficiently is to observe that the number is close to 1/5.

If $a/b \approx .202532$ then $0 < a/b - 1/5 \approx .002532$.

Hence, multiplying through by 5b,

$$0 < 5a - b \approx .012660\, b < 2 \text{ (since } b < 100).$$

As the integer $5a - b$ lies between 0 and 2, we must have

$$5a - b = 1, \text{ and } b = 5a - 1.$$

Substituting back in

$$\frac{a}{b} = \frac{a}{5a-1} \approx .202532$$

and solving we obtain $a = 16$, $b = 79$.

<u>0104</u> In a certain community there are a number of clubs, none of which includes the entire community in its membership.(1) For every pair of distinct clubs there are at least two people who are members of each.(2) No two clubs have exactly the same members;(3) if any three people are considered

0104 Cont. there is one and only one club having all three as members.(4) If there are at least two clubs(5) show that -

(a) for every pair of clubs there are exactly two people who are members of both;

(b) the community contains at least four people;

(c) every club has a membership of at least three.

Can you prove anything else about the clubs?

Answer:

(a) follows easily from (2) and (4);

(b) From (5) there exist at least two clubs X and Y. By (a) there are two people, A and B, who belong to both. By (3) at least one of X and Y must contain a third member C; let X be that club which contains C. By (1) there is some member of the community D, not in X. Hence (b) is proved.

(c) It is immediately clear from (2) that every club has at least two members. Suppose the club Y has only two members A and B. If C is a third member of the community there exists (by (4)) a club X containing A, B and C. By (1) there is a person D not in X, and by (4) there exists a club Z having A, B and D as members.

Z cannot also contain C since then Z and X would both contain A, B and C contradicting (4). Now consider the club W containing A, C and D. It cannot also contain B (for then W and X, or W and Z would contradict (4)). But now it appears

cont...

0104 that the clubs Y and W have only member A in
Cont. common, contradicting (2). Hence it is impossible for there to exist a club with only 2 members, and (c) is proved.

....

More theorems about the community and its club structure are possible but we shall leave further discussion to the future to give students the opportunity to discover them.

0105 A new piece has been invented for use in the game of chess. This piece, the duke, makes moves rather similar to those made by the knight. We call the square in the r-th row and the s-th column of the chess board (r, s); a duke situated in this square may move to any of the eight squares $(r \pm 3, s \pm 1)$ or $(r \pm 1, s \pm 3)$ which are within the chess board.

If the duke is in the nearest left-hand corner, i.e. in square (1,1) and there are no other pieces on the board, is it possible to move it to any of the other corner squares (1,8) (8,8) or (8,1)? If so state in each case (a) the minimum number of moves; (b) whether the total number of moves to do this is restricted in any way.

Prove the results you have stated.

Cont.

Answer:

0105 The duke can be moved from (1,1) to (8,8) but not to either (1,8) or (8,1). The minimum number of moves required to reach (8,8) is 5. More moves may be used but the number required is invariably an odd number. These statements are easily proved by noticing that at each move the parity (oddness or evenness) of both co-ordinates of the square occupied by the duke changes. If the duke starts at (1,1) with both co-ordinates odd, then after 1 move it will reach a square with both co-ordinates even; on the second move it will reach a square with both co-ordinates odd, and so on. It is clear that it can never reach a square with one co-ordinate odd, and one even, such as (1,8) or (8,1). Also after an even number of moves both co-ordinates will again be odd. Hence if it can reach (8,8) at all it requires an odd number of moves.

The sum of the two co-ordinates can increase by a maximum of $3 + 1 = 4$ at any move, and since to get from (1,1) to (8,8) an increase of 14 must be obtained - more than 3 moves, and hence at least 5 must be needed. That this is sufficient is seen by the sequence

$$(1,1) \to (2,4) \to (5,5) \to (6,2) \to (7,5) \to (8,8)$$

0106 The function $P^{(d)}(x,y,z)$ where d is a positive integer, is the sum of all the distinct products $x^a y^b z^c$ where a, b, c are positive integers such that $a + b + c = d$. For example

$$P^{(1)}(x,y,z) = x + y + z, \text{ and}$$

Cont.

0106 $P^{(2)}(x, y, z) = x^2 + y^2 + z^2 + yz + zx + xy.$

If $x = 1$ and $y > 0$, $z > 0$ show that

$P^{(d+1)}(x, y, z) \quad P^{(d)}(x, y, z).$

If $x = 1$, $y = \frac{1}{4}$, and $z = \frac{1}{9}$ show that

$P^{(d)}(x, y, z) \quad 1.5$ for all finite d.

<u>Answer:</u> We extend the notation so that $P^d(y, z)$ is the sum of all distinct products $y^b z^c$ where $b\,c$ are non-negative intergers such that $b + c = d$.

Then $P^{d+1}(x, y, z) = x P^d(x, y, z) + P^{d+1}(y, z)$

from which the first statement to be proved is obvious.

Also $P^d(x, y, z) = x^d + x^{d-1}P^1(y, z) + x^{d-2}P^2(y, z) + \ldots$

$+ x^1 P^{d-1}(y, z) + P^d(y, z)$

$$P^d(y, z) = y^d + y^{d-1} z + \ldots + yz^{d-1} + z^d$$

(a G. P with ratio z/y)

$$= \frac{(y^{d+1} - z^{d+1})}{(y - z)}$$

Hence $P^d(1, y, z) = 1 + \frac{1}{y - z}\left[(y^2 - z^2) + (y^3 - z^3) + \ldots\right.$

$\left.\ldots + (y^{d+1} - z^{d+1})\right]$

$$= \frac{1}{y-z}\left[\frac{y - y^{d+2}}{1 - y} - \frac{z - z^{d+2}}{1 - z}\right]$$

Cont. .

O106.(cont.)

$$= \frac{1}{y - z}\left[\frac{y - z}{(1 - y)(1 - z)} - \left(\frac{y^{d+2}}{(1 - y)} - \frac{z^{d+2}}{(1 - z)}\right)\right]$$

$$= \frac{1}{(1 - y)(1 - z)} - R$$

$$= 1 \cdot 5 - R \quad \text{where } R > 0.$$

Cutting the Cake. (p 31)

Answer. One method consists in moving a knife slowly over the cake, keeping its direction constant. Each of the n peopleamongst whom the cake is to be divided may at any time call for the knife to be stopped, and when this happens the cake is sliced vertically at this point, and the piece is given to this person. It is clear that he believes that he has received 1/n th of the cake, or he would not have spoken, whilst the others cannot believe that he has received more, since anyone so judging would have spoken earlier to protect his own interests. Continuing in this way, the cake can be completely divided amongst all without any discontent. (If several people speak simultaneously, the slice may be given to any one of the group).

Other methods of proceeding are known.

Duke's Tour. (See p 12)

Wormald's Solution.

(1, 1) - (2, 4) - (3, 1) - (6, 2) - (3, 3) - (2, 6) - (1, 3) - (4, 2) - (7, 1) - (8, 4) - (7, 7) - (4, 8) - (1, 7) - (4, 6) - (1, 5) - (2, 8) - (3, 5) - (2, 2) - (5, 3) - (8, 2) - (5, 1) - (6, 4) - (5, 7) - (8, 6) - (5, 5) - (6, 8) - (3, 7) - (6, 6) - (7, 3) - (4, 4) - (7, 5) - (8, 8).

0107 Sketch the curve $y = x^{-1/2}$ between $x = 0$ and $x = 1$ and calculate

$$\int_0^1 x^{-1/2} dx.$$

Using the above result, or otherwise, show that if n is a positive integer, the expression

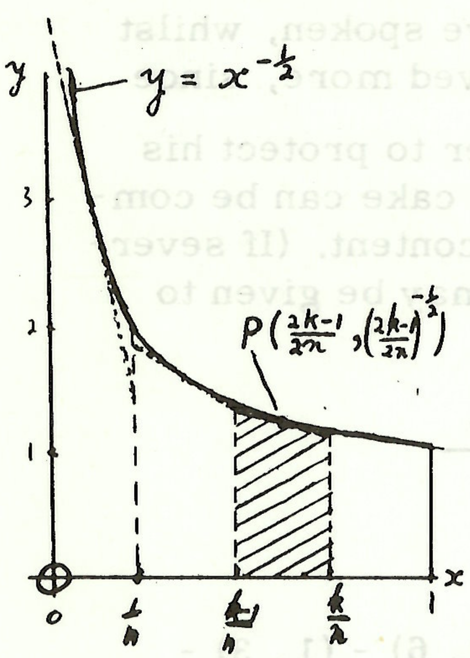
$$\frac{\sqrt{2}}{\sqrt{n}} \left\{ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} + \dots \frac{1}{\sqrt{2n-1}} \right\}$$

lies between 1 and 2.

Answer

$$\int_0^1 x^{-1/2} dx = \lim_{A \to 0} \int_0^1 x^{-1/2} dx$$

$$= \lim_{A \to 0} \left(2.1^{1/2} - 2.A^{1/2} \right)$$



Let the interval [0, 1] on the x axis be divided into n equal intervals, The kth such interval is

$$\left(\frac{k-1}{n}, \frac{k}{n} \right)$$

and its mid point is $\frac{2k - 1}{2n}$.

The tangent at the point $P\left(\frac{2k-1}{2n}, \left(\frac{2k-1}{2n} \right)^{-1/2} \right)$ lies below the graph of $x^{-1/2}$ since this is concave upward (its second derivative is positive) and therefore the area under this tangent line in the strip

$$\frac{k-1}{n} \leq x \leq \frac{k}{n}, \quad y \geq 0$$

is less than the area under the graph in this strip. (See the diagram where n = 4 and the shaded region is the third strip).

The area of the trapezium under the tangent line is, by mensuration,

$$\text{width x average height} = \frac{1}{n} \times \left(\frac{2k-1}{2n}\right)^{-1/2}.$$

Therefore

$$\frac{1}{n} < \frac{1}{n}\sqrt{\frac{2n}{2k-1}} < \int_{\frac{k-1}{n}}^{\frac{k}{n}} x^{-1/2}\,dx. \quad k = 1, 2, \ldots. n.$$

Summing for the n different values of k gives

$$n.\frac{1}{n} < \sqrt{\frac{2}{n}}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} + \ldots.. + \frac{1}{\sqrt{2n-1}}\right) <$$

$$\int_0^1 x^{-1/2}\,dx = 2.$$

0108 Prove that if the integers $a_1, a_2, \ldots a_n$ are all distinct, then the polynomial

$$(x-a_1)^2(x-a_2)^2(x-a_3)^2 \ldots (x-a_n)^2 + 1$$

cannot be written as the product of two other polynomials with integral coefficients.

Answer:

Suppose, on the contrary, there exist polynomials $p(x)$, $q(x)$ with integer coefficients such that

$$p(x)\, q(x) = (x-a_1)^2(x-a_2)^2 \ldots (x-a_n)^2 + 1 . \qquad (1)$$

Since the right hand side is always positive $p(x)$ can never vanish, and hence its sign never changes. We may assume that $p(x)$ and $q(x)$ are positive for all real x. Substituting $x = a_k$ in (1) gives

$$p(a_k)\, q(a_k) = 1 \quad \text{whence}$$

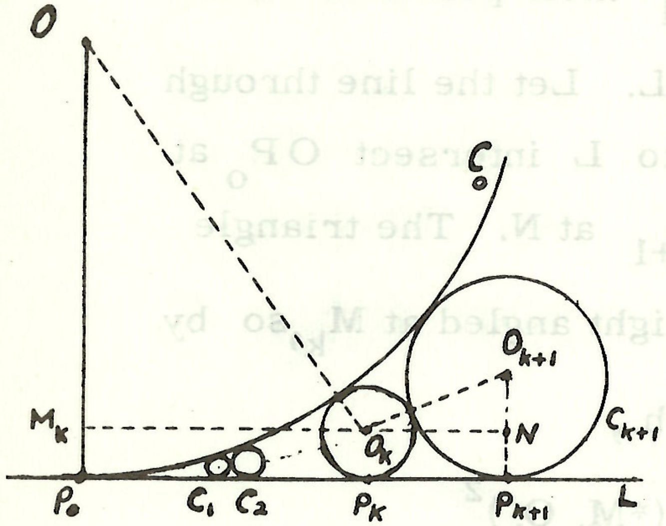
$$p(a_k) = q(a_k) = 1 \quad \text{for } k = 1, 2, \ldots n.$$

By the factor theorem $p(x) = 1 + F(x)(x-a_1)\ldots(x-a_n)$

and $q(x) = 1 + G(x)(x-a_1)\ldots(x-a_n)$.

where $F(x)$ and $G(x)$ are polynomials. From (1) the degree of $p(x)\,q(x)$ is $2n$ so that $F(x)$ and $G(x)$ both have degree 0, i.e. they are constants, say $F(x) = a$, and $b(x) = b$. Substitution in (1) now gives $ab = 1$ and $a + b = 0$, but there are no real numbers a and b satisfying these equations.

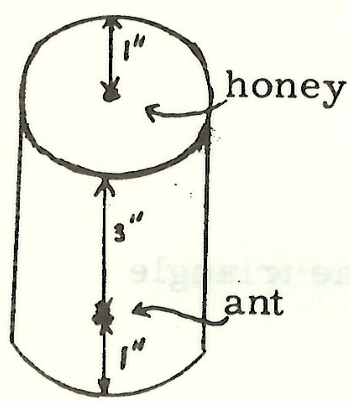
0109



C_o is a circle of diameter D, touching a line L. The circles $C_1, C_2, C_3, \ldots, C_n$ are constructed in succession with the line L, each $C_i (i > 1)$ touching C_o, L, and C_{i-1}. (C_1 is any circle touching both L and C_o.) If the diameter of C_i is equal to D_i, find a formula for D_n in terms of D and D_1.

(Solution next page)

The Ant and the Honey.



The diagram shows a perfectly cylindrical can four inches high and six inches in circumference. On the outside, one inch from the bottom, is an ant. On the inside, one inch from the top, and directly opposite, is a spot of honey. What is the shortest distance which must be walked by the ant to reach the honey?

(Answer p. 32)

0109 (Answer) Let O, O_k, O_{k+1} be the centres of C, C_k and C_{k+1} respectively and P_o, P_k, P_{k+1} their points of contact with the line L. Let the line through P_k parallel to L intersect OP_o at M_k, $O_{k+1} P_{k+1}$ at N. The triangle $OM_k O_k$ is right angled at M_k, so by Pythagoras Th.,

$$(*OO_k)^2 = (*OM_k)^2 + (*M_k O_k)^2$$

i. e.

$$\left(\frac{D}{2} + \frac{D_k}{2}\right)^2 = \left(\frac{D}{2} - \frac{D_k}{2}\right)^2 + \left(*P_o P_k\right)^2$$ and hence

$$D. D_k = \left(*P_o P_k\right)^2 \qquad (1)$$

Similarly

$$D. D_{k+1} = \left(*P_o P_{k+1}\right)^2 \qquad (2)$$

Again applying Pythagoras Thm. to the triangle $O_k N O_{k+1}$,

$$\left(*O_k O_{k+1}\right)^2 = \left(*O_k N\right)^2 + \left(*NO_{h+1}\right)^2 .$$

$$\left(\frac{D_k}{2} + \frac{D_{k+1}}{2}\right)^2 = \left(*P_oP_{k+1} - *P_oP_k\right)^2 + \left(\frac{D_{k+1}}{2} - \frac{D_k}{2}\right)^2$$

Substituting for $*P_oP_k$ and $*(P_oP_{k+1})$ from (1) and (2) and simplifying gives

$$\sqrt{D_{k+1}} = \frac{\sqrt{D}\ \sqrt{D_k}}{\sqrt{D} - \sqrt{D_k}} \qquad (3)$$

Putting $k = 1$

$$\sqrt{D_2} = \frac{\sqrt{D}\ \sqrt{D_1}}{\sqrt{D} - \sqrt{D_1}} \qquad (4)$$

then for $h = 2$,

$$\sqrt{D_3} = \frac{\sqrt{D}\ \sqrt{D_2}}{\sqrt{D} - \sqrt{D_2}}$$

and substituting for D_2 using (4) gives

$$\sqrt{D_3} = \frac{\sqrt{D}\ \sqrt{D_1}}{\sqrt{D} - 2\sqrt{D_1}}$$ after simplifying.

This suggests the general formula

$$\sqrt{D_{k+1}} = \frac{\sqrt{D}\ \sqrt{D_1}}{\sqrt{D} - k\sqrt{D_1}} \qquad (5)$$

which we prove by induction.

Assuming $\sqrt{D_k} = \dfrac{\sqrt{D}\ \sqrt{D_1}}{\sqrt{D} - (k-1)\sqrt{D_1}}$

we have using (3) $\sqrt{D}_{k+1} = \dfrac{\sqrt{D}.\ \dfrac{\sqrt{D}\ \sqrt{D}_1}{\sqrt{D} - (k-1)\sqrt{D}_1}}{\sqrt{D}.\ - \dfrac{\sqrt{D}\ \sqrt{D}_1}{\sqrt{D} - (k-1)\sqrt{D}_1}}$

$$= \frac{\sqrt{D}\ \ \sqrt{D}_1}{(\sqrt{D} - (k-1)\ \sqrt{D}_1) - \ D_1}$$

which gives (5).

Since (5) is true when $k = 1$ and is true for $k + 1$ if it is true for k, it follows that it is true for all positive integers.

<u>0110</u> Let N be an arbitrary natural number. Prove that there exists a multiple of N which contains only the digits O and 1. Moreover, if N is relatively prime to 10 (that is, is not divisible by 2 or 5), then some multiple of N consists entirely of ones. (If N is not relatively prime to 10, then, of course, there exists no number of the form 11 ... 1 which is divisible by N.)

<u>Answer:</u> We shall do the second part first. Suppose N is relatively prime to 10, and put $M = 9N$. Let r_k be the remainder when 10^k is divided by M. Now $r_1, r_2, r_3 \ldots, r_M$, are all positive integers less than M so that at least one pair are equal.

Suppose $r_s = r_t$ and $s < t$.

$$10^s = q_1 M + r_s$$

$$10^t = q_2 M + r_t \ .$$

Subtracting, $10^s(10^{t-s} - 1) = (q_2 - q_1)M$

and it follows, since M is relatively prime

to 10^s that M is a factor of $10^{t-s} - 1$.

But $10^{t-s} - 1 = 9\,9 \ldots 9$ with (t-s) nines, and it is immediately clear that N is a factor of $1\,1 \ldots 1$ with (t-s) ones.

Now the first part is easy. If N is any integer it can be factorized as

$N = 2^{\alpha} \; 5^{\beta} K$ where K is relatively prime to 10. By the above working there exists an integer L such that

$$LK = 1\,1 \ldots 1.$$

If γ is the larger of α and β, N is a factor of

10^{γ} LK a number whose decimal representation contains only a string of ones followed by a string of γ noughts.

Cutting the Cake. If two people have a cake to be divided between themselves, arguments as to the fairness of the division can be avoided by letting one of them cut it, on the understanding that the other has first choice. What procedure can be adopted if the cake is to be shared amongst n people, to ensure that all are satisfied with the division.?

(Answer p 23)

Answers to the puzzles.

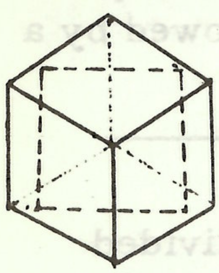
Dudeney's Map-Folding Puzzle. (p.16)

(1) Fold the R. H. end under so that 4 is beneath 3, and 5 beneath 6. Then fold the lower three squares under the upper three, i. e. fold 2 beneath 1, etc. Now fold the pile with 3 on top under that with 8 on top, and finally the resulting pile under that with 1 on top.

(2) Fold the R. H. half under, so that 4 is beneath 1, 5 beneath 2, etc. Then fold the lower two squares under the upper two, i. e. fold 3 under 8, and 2 under 1. The end of the middle two leaves, consisting of the squares 4 and 5 must now be folded over and tucked between the squares 3 and 6. Lastly, fold the squares 1 and 2 on top of the remaining pile.

(3) Fold the lower four squares under the upper four, so that 4 is beneath 1, etc. Bend the resulting strip round into a circle, and insert the (7, 6) pair between the (1, 4) pair like a snake starting to eat itself. Continue the eating process until the square labelled 2 is underneath that labelled 1.

Puzzles with cubes. (p. 8)



(1) It is possible. If the smaller cube is view- along a diagonal, one sees at once that a square hole larger than the face of the cube can be made through it, parallel to the diagonal. See the diagram.

(2) The resistance of the network is 5/6 ohms.

The Ant and the Honey. (p 27)

The shortest distance the ant must walk is five inches.