GAMES of STRATEGY

The playing of games has its origins in antiquity - the ancient Romans, Greeks, and Chinese all played games of varying degrees of difficulty. The characteristic feature of all games is that they involve a certain amount of conflict between the participants. This conflict arises from the basic fact that each player wishes to win the game, and win by as much as possible.

The games we are all familiar with fall into two broad categories.

1. Games of Chance - in which the players have no control over the choice of moves or over the outcome of the game. All the moves are chance moves whose probabilities of occurrence are known. In fact, the theory of probability originated in an attempt by Pascal to analyse a game of pure chance, this analysis resting on the notion of expected value. Suppose n mutually exclusive events are possible at each move and the probabilities of these events are $p_1, p_2, \ldots, p_n$, and $a_1, a_2, a_3, \ldots, a_n$ are the respective amounts a player receives in case of their occurrence. Then the sum $a = p_1a_1 + p_2a_2 + \ldots + p_na_n$ is called the expected value for the player. It represents the fair way to estimate the stake of the player before the occurrence of the event. Two-up and Snakes and Ladders are examples of games of chance.

2. Games of Strategy - or games of "skill". In these games a certain amount of control over the outcome of the game can be exercised by each player. At the same time, in most of these games, the element of chance is not excluded. Examples of this type of game are Bridge, Chess, and

Poker. The precise study of such games has been described formally as "a mathematical theory of decision making by participants in a competitive environment". With this definition we may (and do) consider Economics and the manoeuveringof military forces as games.

In 1921 the Frenchman Emile Borel made the first attempt to describe games mathematically, and in 1928 the first great advance in this theory was made by the brilliant Hungarian mathematician John von Neumann. The theory did not receive much attention until the appearance, in 1944, of the definitive study "The theory of games and economic behaviour" written by von Neumann and Oskar Morgenstern.

We would now like to classify games of strategy. Firstly we may distinguish between games according to the number of players. For example, Patience is a 1-person game.; Chess a 2-person game. Bridge is also a 2-person game, as North and South have identical interests, and collaborate as one team. Similarly East and West may be considered as one person. A study of economics in Australia could be considered as a 12,000,000-person game. Secondly, consider the sordid question of money. The large majority of games (including economics) are conducted in such a way that after each play, each participant pays each other participant an amount (possibly zero or negative) of money. If a 2-person game is such that one player's winnings are the other player's losses we call the game a <u>zero-sum</u> game. Most parlour games, such as Bridge, are zero-sum games. However, any reasonable economic theory will be a non-zero-sum game as economic processes usually create (or destroy) wealth.

A game is <u>finite</u> if it has only a finite number of moves, each of which involves a finite number of alternatives. Chess is a large, but nonetheless finite game. An example of an infinite game is that in which each player picks a natural number , the one with the larger winning the difference between the numbers.

Finally, games may be classified according to the amount of information available to each player regarding the past and future choices of the other player. A game with <u>Perfect information</u> is one in which the players are given complete information about all previous moves, be they strategic or chance moves. For example, Chess, Draughts, and Go, are perfect information games, whereas Poker and Bridge are not.

We are now in a position to study games from a mathematical viewpoint. We shall confine our attention to 2-person zero-sum games, of which we first present some simple examples.

(1) A board is divided into 4 equal squares and in each square a number is placed (see fig.). The two players, call them P and Q, move as follows: P chooses a row of the board (e.g. he chooses row 1), and Q, without being informed of P's choice, chooses a column (e.g. column 2). The number lying in P's row and Q's column is the amount Q pays P (in this case Q pays P 10).

-5	10
3	-4

(2) Two fingered Morra. (This game was played in ancient Rome).

Each player simultaneously show either 1 or 2 fingers, and at the same time guesses the number of fingers his opponent will show. If only one player guesses correctly, he receives an amount equal to the total number of fingers shown by himself and his opponent; in all other cases the game is drawn.[1] As in the previous game, Morra may be represented by a rectangular array, viz.

		Q's possibilities			
		(1,1)	(1,2)	(2,1)	(2,2)
P's Possibilities	(1,1)	0	2	-3	0
	(1,2)	-2	0	0	3
	(2,1)	3	0	0	-4
	(2,2)	0	-3	4	0

Here a pair such as (2,1) is a possible move for Q; meaning that he shows 2 fingers and guesses 1. Thus if Q shows(2,1) and P chooses (2,2), Q pays P 4. Similarly if Q chooses 2 fingers and guesses 2, whilst P chooses 2 fingers and guesses 1, P pays Q 4.

(3) Consider the game in which P chooses a row from the array.

$$\begin{bmatrix} 2 & 1 & 10 & 11 \\ 0 & -1 & 1 & 2 \\ -3 & -5 & -1 & 1 \end{bmatrix}$$

and Q chooses a column. The element common to both the chosen row and column is the payoff from Q to P.

1) It has been said that an honest man is one with whom you could play Morra in the dark.

Any 2-person, zero-sum finite game can be thought of as a game of the above kind. A <u>strategy</u> for a player is a plan, formulated before the play, for playing the game. This plan should cover all possibilities which may arise and must include all information which may become available during the game. Suppose P has m different strategies labelled 1, 2, .., m; Q has m strategies 1, 2,.., m. On the first move P chooses strategy i say, and on the second move Q (without being informed of P's choice) chooses strategy j say; and suppose that Q then pays P an amount a (i, j) or equivalently P pays Q (-a(i, j)). The game is then completely determined by P's <u>payoff matrix</u>[1])

$$M = \begin{bmatrix} a(1,1) & a(1,2) - & - & - & - & a(1,n) \\ a(2,1) & a(2,2) - & - & - & - & a(2,n) \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a(m,1) & a(m,2) - & - & - & & a(m,n) \end{bmatrix}$$

In this matrix, the second strategy of P (for example) is represented by the row (a(2,1), a(2,2), -, a(2,n)) and Q's first strategy by the column

$$\begin{Bmatrix} a(1,1) \\ a(2,1) \\ \\ -- \\ -- \\ a(m,1) \end{Bmatrix}$$

so that if these are the respective strategies chosen the payoff from Q to P is a(2,1).

If P chooses strategy i and Q chooses strategy j, then P wants a (i, j) as large as possible - but he controls only the choice of i. Similarly Q wants a(i, j) as small as possible- and he controls only the choice of j. We now ask the basic

1)A matrix is simply an ordered array of numbers.

question of game theory: what guiding principles are there which should determine the choices of strategy for each player? In particular, is there an optimal way of playing the game? That is, can one give rational arguments in favour of playing the game one way rather than another?

In special cases this question can be easily answered. Consider game (3) above. Each element of the first row of the payoff matrix is greater than the corresponding element of both row 2 and row 3. So P will do best (regardless of what Q does) if he always chooses row 1 as his strategy- this then is the optimal way for P to play. Similarly, each element in column 2 is smaller than the corresponding elements in the other columns. Thus Q's optimal strategy is to always play column 2; any other strategy on Q's part would be, by definition, a poor strategy.

Let us return to the general payoff matrix M. If P chooses strategy i, he must be paid at least the smallest of the numbers in the ith row; i.e., he must be paid at least $\min_{j=1,2,\ldots,n} a(i,j) = \min_j a(i,j)$. Since P can choose i at will, he can choose it so as to make $\min_j a(i,j)$ as large as possible. So there is a choice i for P which ensures that he gets at least

$$\max_i \min_j a(i,j).$$

In a similar way (remembering the game is 0-sum), there is a choice of strategy j for Q which ensures that he gets paid at least $\max_j \min_i (-a(i,j))$. That is, so that P gets paid no more than $\min_j \max_i a(i,j)$. It can be shown that

$$\max_i \min_j a(i,j) \leq \min_j \max_i a(i,j);$$ if equality occurs, i.e.: if

(+) $\max_i \min_j a(i,j) = \min_j \max_i a(i,j) = v$

then P must realize that he can get v, and that he can be prevented from getting more than v by his opponent. So, unless he has some good reason for believing that Q will do something wild, P might as well settle for v as his payoff, and play to get it.

Similarly Q might as well settle for (-v) as his payoff. In this case there are strategies I, J for P and Q respectively such that

$$a(I, J) = v; \text{ and}$$

$$a(i, J) \leqslant a(I, J) \leqslant a(I, j) \quad \text{for all } i \text{ and } j.$$

I and J are then the optimal strategies for P and Q respectively, as they have the following properties:

(i) If P chooses I, then no matter what strategy Q chooses, P must be paid at least v;

(ii) If Q chooses J then no matter what strategy P chooses, P cannot be paid more. than v ;

(iii) If P announces in advance that he will play strategy I, Q cannot use this information to reduce P's payoff.

If condition (+) is satisfied, v is called the value of the game and it represents the amount P should pay Q at the beginning of the game in order to equalize the winnings in the game. The payoff matrix M is said to have a saddle point at I, J.

A few illustrations of what we have done may help the reader to understand the subject.

The game with payoff matrix

$$\begin{bmatrix} 12 & 13 & 12 \\ 10 & 31 & 9 \end{bmatrix}$$

has two saddle points at (1, 1) and at (1, 3) - the value of the game being 12. So P's optimal strategy is to always choose row1; Q's optimal strategy is always to choose column 1 (or column 3).

So if a game has a saddle point it is a simple matter (in theory at least) to find the optimal strategies for each player - all we have to do is look for an element of the payoff matrix which is both the minimum of the row it is in and the maximum of the column it is in. Now it can be shown that a game with perfect information does have at least one saddle point so that one can find optimal strategies for these games. As observed previously chess is a game with perfect information. Thus to find the optimal strategy in a game of chess we form the payoff matrix, (consisting of N rows and N columns where N is the total number of strategies available !) whose entries consist of +1; 0 or (-1) depending on whether P wins; the game is a stalemate or draw, or Q wins. To find the optimal strategy for P we find a saddle point of this matrix. Of course such a saddle point has never been found because of the immense size of the payoff matrix.

Having thus dispensed with games with saddle points, we turn to those games whose payoff matrix does not have a saddle point. One such matrix is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} ,$$

as in this case $-1 = \max_i \min_j a(i,j)$, $\min_j \max_i a(i,j) = +1$

What strategy should P adopt in playing this game? First of all, it is obvious that it makes no difference whether P chooses row 1 or row 2, as in either case he will receive +1 or -1 according as Q makes the same or different choice. However, if Q <u>knows</u> what choice P will make, then Q can ensure (by making the opposite choice) that P will have to pay him 1. Hence P is at a distinct disadvantage if Q discovers his strategy. So it is of the utmost importance to P that he prevent Q from making this discovery, and the best way of ensuring this is to arrange the situation so that <u>even P himself</u> does not know what strategy he will play in advance of the play! One way to do this is to decide what play to choose by means of some chance device. In this case P could throw a coin, choosing strategy (or row) 1 if a head shows and strategy 2 if a tail shows.

Obviously Q should also adopt this method of play. Thus the probability that P chooses row 1 is $\frac{1}{2}$, and we observe that the mathematical expectation of P, whatever Q does, is $1 \cdot (\frac{1}{2}) + (-1) \cdot (\frac{1}{2}) = 0$. In fact this is the only way P can play the game without running the risk of losing if Q discovers what he (P) is going to do.

Let us return to the game mentioned in the beginning whose payoff matrix was

$$\begin{bmatrix} -5 & 10 \\ 3 & -4 \end{bmatrix}$$

Since the game has no saddle point it would seem desirable that both P and Q play the game using chance devices to decide on their choice of strategy. Suppose P decides to play strategy 1 with probability x, so he plays 2 with probability 1-x. Similarly, Q plays 1 with probability y, and 2 with probability 1-y. The expected payoff to P is then:

$$\begin{aligned} E(x, y) &= -5xy + 10\,x(1-y) + 3y(1-x) - 4(1-x)(1-y) \\ &= -22\,x\,y + 14x + 7y - 4 \\ &= -22 \left\{x - \frac{7}{22}\right\}\left\{y - \frac{14}{22}\right\} + \frac{5}{11}. \end{aligned}$$

Thus if P takes $x = \frac{7}{22}$, he ensures that his expectation will be at least $\frac{5}{11}$. Moreover, he cannot be sure of more than $\frac{5}{11}$; for, by choosing $y = \frac{14}{22}$, Q can ensure that P's expectation will be exactly $\frac{5}{11}$. Indeed, by choosing y such that $\left\{y - \frac{14}{22}\right\}$ and $\left\{x - \frac{7}{22}\right\}$ have the same sign, Q can even reduce P's expectation below $\frac{5}{11}$. So P might as will settle for $\frac{5}{11}$ and play row 1 7 times in 22, and row 2 15 times in 22.

Similarly, Q might as well settle for $-\frac{5}{11}$ and play his first strategy 7 times out of 11 and his second strategy 4 times out of 11. It now seems reasonable to call these methods of play the optimal strategies for P and Q, and to call $\frac{5}{11}$ the value of the game.

Consider now the general payoff matrix M, and suppose it has no saddle point. By a mixed strategy for P, we shall mean an ordered collection $(x_1, x_2, .., x_m)$ of non-negative real numbers satisfying the condition $x_1 + x_2 + .. + x_m = 1$. The ith element x_i of this collection represents the probability that P chooses the ith row as his strategy on a given move. The strategies considered in the previous games where there was a saddle point were also of this kind - as there we always chose one row, say row k, and this is equivalent to playing the mixed strategy $(x_1, x_2, .., x_k, .., x_m)$ where $x_k=1$ and $x_i = 0$ if $i \neq k$. This type of strategy is called a pure strategy.

Suppose then that P plays the mixed strategy $X = (x_1, ..., x_m)$ while Q uses the mixed strategy $Y=(y_1, .., y_n)$. The mathematical expectation of P is then

$$E(X, Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} a(i,j)\, x_i\, y_j.$$

If for some X* and some Y*, we have

$$(++) \qquad E(X, Y^*) \leqslant E(X^*, Y^*) \leqslant E(X^*, Y).$$

for all mixed strategies X and Y for P and Q respectively, then we call X* and Y* optimal(mixed) strategies for P and Q., and we call E(X*, Y*) the value of the game. By using X* P ensures that he will receive at least E(X*,Y*) regardless of what Q does; similarly, by using Y*, Q can keep P from getting more than E(X*, Y*). Thus E(X*, Y*) is the amount P can reasonably expect to get (he can get it by playing X*) and Q can hold him down to it by playing Y*.

If it happens that both the numbers

$$v_1 = \max_X \min_Y E(X, Y)$$

and

$$v_2 = \min_Y \max_X E(X, Y)$$

are equal, then there exist mixed strategies X*, Y* satisfying (++) - so in this case the game has a solution.

It can be shown [1] that for any 2-person, zero-sum, finite game v_1 is equal to v_2 - this is a statement of the fundamental result of game theory, the MINMAX theorem.

Using the MINMAX theorem, we see that there is an optimal strategy for the game of 2-fingered Morra (in fact there are many such strategies for this game). The particularly clever reader can show by first of all observing that the value of the game is 0, that this optimal mixed strategy is $(0, \frac{3}{5}, \frac{2}{5}, 0)$ for both players. So an optimal way of playing 2-fingered Morra is as follows: roll a dice, if either a 1, 2 or 3 appears, show 1 finger and guess 2 ; if a 4 or 5 appears, show 2 fingers and guess 1; if a 6 appears, roll again.

The mathematical theory of games has been developed to a stage where many of the more difficult games can be studied quite successfully. For example, n-person games and infinite games are known to have optimal strategies in general; continuous games (in which the sums above are replaced by integrals) may be analyzed via a MINMAX theorem similar to the one we have seen. The theory does not yet extend to non zero-sum games [2] In fact, very little is known about such games. These are just the games which would be important in any study of economics based on game theory. Even if such a theory could be developed, the economic structure of our society is so involved that the theory could only hope to give quite vague lines for economists to follow. Conceivably, the theory of games could be put to some use in competitive society.

1) The proof involves a study of the geometry of n-dimensional space - this includes the algebra of matricies.

2) A typical example of a non-zero sum game is Russian roulette.

Consider, for example, the situation in the U. S. A. where there are effectively only 2 mail-order companies - so that we have a 2-person, zero-sum game. An example in which competition arises is the timing of the publication of the winter catalogue. If one company distributes its catalogue before that of its competitor, it is presumed that the cream of the buying will be captured. On the other hand, if this policy is pursued too far, the catalogue loses its effectiveness because it attempts to sell woolen underwear and snow shovels in the middle of summer. Actual practice has not, however, settled down to an optimum but to a combination of loss from both scores - publication at the same time, moreover early in June- just before summer! Game theory indicates a MINMAX solution of the mixed strategy - a solution which is unlikely to be adopted by either company!

The author of the above article, Dr J. D. Gray, a lecturer at the University of New South Wales, is an expert in an important field of mathematics known as functional analysis. He received his Ph. D. degree from the U. of N. S. W. earlier this year after study and research in both America and Australia.

Correct Solutions

The following students submitted correct solutions of the indicated problems set in the last issue of Parabola.

M. Doyle (St Joseph's College, Hunter's Hill) J111.
R. Sebesfi (St Joseph's College) J111
R. Smykowsky (St Joseph's College) J111
D. Nash (Parkes H. S.) O113, O115, O116 and O120
J. Armstrong (St Joseph's College) O117 and O120
S. McHale (St Joseph's College) O117 and O119
D. McKenzie (St Joseph's College) O117 and O120.