SCHOOL MATHEMATICS COMPETITION

Junior

1 (i) a, b are integers such that a + b and $a^2 + b^2$ are both divisible by 7. Prove that a and b are both divisible by 7.

1 (ii) Is the statement still valid if each 7 is replaced by 49?

Answer (i) Both $a^2 + b^2$ and $(a + b)^2$ are divisible by 7; hence, their difference, 2ab, is divisible by 7. If 2ab is decomposed into its prime factors, 7 must occur, hence it is a factor of either a or b. But since it is also a factor of their sum it must divide both.

Answer (ii) The above argument does not apply for the number 49 since it is not a prime. In fact the statement is not true, since if a = 42, b = 7, both (a + b) and $a^2 + b^2$ are divisible by 49 but neither a nor b is.

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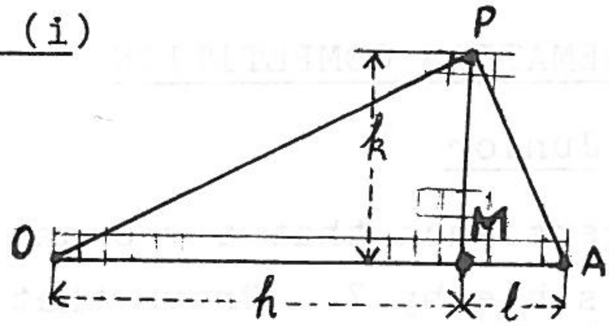
2 (i) A rectangular room is paved with square tiles of the same size. Show how to draw a right-angled triangle such that:

a) the vertices of the triangle are on corners of the tiles;
b) the hypotenuse lies along the wall of the room;
c) the ratio of the two smaller sides is 2:3.

2 (ii) Can you generalize for an arbitrary ratio m:n?

Answer (i)



Let OPA be such a triangle, with OM* = h units, MA* = l units and MP* = k units (l unit is the length of the side of a tile)

Then (by elementary trigonometry or by using similar triangles)

$$\frac{AP^*}{OP^*} = \frac{PM^*}{OM^*} = \frac{AM^*}{MP^*} \text{ (= tan } \theta)$$

$$\frac{2}{3} = \frac{k}{h} = \frac{l}{k} \text{ whence } l:k:h = 4:6:9$$

Thus the smallest such triangle has l = 4, k = 6, and h = 9.

Answer (ii) Similar working gives

$$\frac{m}{n} = \frac{l}{k} = \frac{k}{h} \text{ whence } l:k:h = m^2:mn:n^2$$

the smallest triangle having $l = m^2$, $k = mn$, and $h = n^2$.

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3 At a party each boy shakes hands with an odd number of girls and each girl shakes hands with an odd number of boys. Show that the total number of children at the party is even.

Answer Let the total number of handshakes (to be precise, boy-girl handshakes) made by all the boys be N. If there are m boys present, N is the sum of m odd numbers so its parity (i.e. oddness or evenness) is the same as m.

Cont.

Now N is also the number of such handshakes made by the n girls present. Therefore by the same argument N and n have the same parity.
Hence m and n are either both odd (if N is odd) or both even (if N is even) and in either case m+n is even.
An alternative wording of this argument is contained in the answer to problem 0117, in this issue.

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4 In the triangle ABC let AB* > AC*, and let D be the foot of the perpendicular from A onto BC.
Prove that:

$$AB^* - AC^* < BD^* - DC^*.$$

AB* denotes the length of AB.

Answer 1

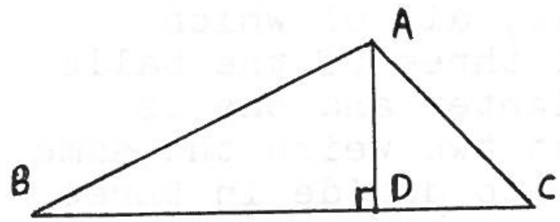
By Pythagoras' Theorem

$$AB^{*2} - BD^{*2} = AC^{*2} - CD^{*2} \; (= AD^2)$$

Hence $AB^{*2} - AC^{*2} = BD^{*2} - CD^{*2}$

$$(AB^* - AC^*)(AB^* + AC^*) = (BD^* - CD^*)(BD^* + CD^*)$$

$$(AB^* - AC^*) = \frac{BD^* + CD^*}{AB^* + AC^*}(BD^* - CD^*)$$



and since BD* + DC* = BC* < AB* + AC* (one side of a triangle is less than the sum of the other two) the first factor on the R.H.S. is less than 1, whence the result.

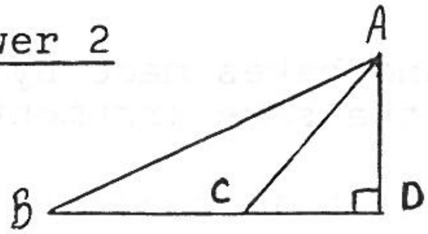
Answer Puzzle: Find the Digits (see p. 16)

A = 1, B = 2, C = 5, D = 3, E = 7, F = 8, G = 6, H = 4.

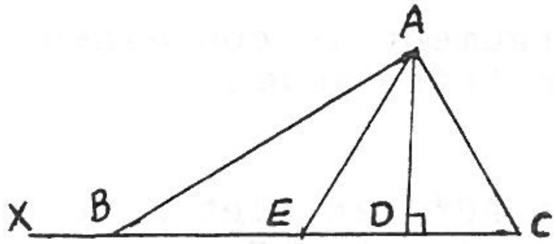
Answer The Frustrated Hiker (see p. 17)

From any point 10 + 5/n.π miles from the south pole. Here n is any positive integer. Also, of course, the north pole is one answer.

Answer 2



If C is an obtuse angle, the result follows immediately since AB* - AC* < BC* by the triangle inequality.



If C is an acute angle, construct E on BC so that DE* = DC*. By congruent triangles ADC and ADE we have AE* = AC*. (Hence AE* < AB* and it is easy to see that E does not lie on CB produced because of the obtuseness of ∠ABX.) In ΔABE we have

AB* < AE* + BE*

AB* - AE* < BE* = BD* - DE* = BD* - CD*.

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5 You are given five golfballs, all of which appear the same. You know that three of the balls have the same weight, one is lighter and one is heavier, but together these last two weigh the same as two regular balls. Show how to decide in three weighings which one is the light ball, which one is the heavy ball and which ones are the regular balls.

The only equipment available is a balance without weights.

Answer 1 Weigh the first ball against the second and then the third against the fourth. Label the fifth ball E. Of the first two weighings at least one pair failed to balance, and possibly both. Hence these balls can be labelled A,B,C and D in such a way that A > B (i.e. A is heavier than B) and either C > D or C = D (i.e. C and D are the same weight).

Cont.

For the third weighing, weigh C against E.

Case 1 If $A > B$ and $C > D$, E must be a regular ball. Hence the third weighing identifies C. If C is regular, D is light and A is heavy. If C is heavy, B is light, and A and D are regular.

Case 2 If $A > B$ and $C = D$, C is regular so that the third weighing identifies E. If E is light, A is heavy and B regular. If E is regular A is heavy and B is light. If E is heavy A is regular and B is light.

Answer 2 Label the balls A,B,C,D and E. Put A and B together in the L.H. scale pan, and C and D together in the R.H. scale pan for the first weighing. For the second weighing, compare A and B, and for the third weighing compare C and D. If $A + B = C + D$ then one scale pan has two regular balls, the other has both the heavy and the light ball. The heavy and light balls are identified in whichever of the second and third weighings does not balance.
If $A + B > C + D$ then the light ball is not on the L.H. pan. If A and B balance they are both regular. If they do not A is heavy and B is regular. Similarly as the heavy ball could not have been on the R.H. pan in the first weighing, C and D are either both regular, or one is regular and one light, and the third weighing shows which situation applies. As A,B,C and D have all been identified, E is also known.
A similar discussion applies if $A + B < C + D$ on the first weighing.

SCHOOL MATHEMATICS COMPETITION

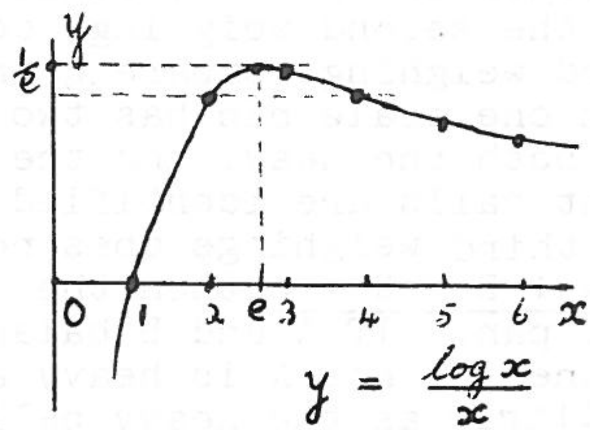
Senior

1 For which positive integers m and n is
$m^n < n^m$?

Answer $m^n < n^m$ ... (1)

Taking natural logarithms, $n \log m < m \log n$ and $\frac{\log m}{m} < \frac{\log n}{n}$. This suggests we discuss $y = \frac{\log x}{x}$ for positive real x. $y' = (1 - \log x)/x^2$. Thus $y' = 0$ at $x = e$, $y = 1/e$; since y' changes +, 0, - as x increases through e, this is a maximum point.

We can produce the rough sketch as shown. (The scales on the two axes are not the same.) Since $y' < 0$ for all $x > e$, y is monotonic strictly decreasing for $x > e$. Hence (1) holds for all cases where $m > n$ and $n \geq 3$.



Now we have to discuss merely the special cases where the positive integers 1 and 2, which are less than e, occur. If $m = 1$, (1) holds for all $n > 1$. As $(\log 2)/2 = (\log 4)/4$, from our first conclusion, (1) holds with $n = 2$ for all $m > 4$. Finally, (1) holds for $m = 2$, $n = 3$. Taking (mn)th roots, from (1) we get that $m^{1/m} > n^{1/n}$. So we could have answered the question equally well by considering $y = x^{1/x}$.

Few candidates solved this problem although many were at least led to state the answer by induction (not mathematical induction) after testing for a few specific values of m and n.

2 (i) a, b are integers and p is an odd prime which divides both $a + b$ and $a^2 + b^2$. Prove that p divides both a and b.

2 (ii) p is a prime greater than 3, and a, b, c are integers such that p divides each of $a + b + c$, $a^2 + b^2 + c^2$ and $a^3 + b^3 + c^3$. Prove that p divides each of a, b and c.

Answer (i) Let $p|x$ mean "p divides x". Since $p|(a+b)$, $p\ (a+b)^2$; thus $p|(a^2+b^2+2ab)$. As $p|(a^2+b^2)$, $p|2ab$. As p is an odd prime, $p|ab$. Hence p divides at least one of a and b. If, for instance, $p|a$, then, as $p|(a+b)$, $p|b$. (Similarly, if $p|b$, we find $p|a$; but, because a and b occur symmetrically in this question, there is no need for us to go through the argument a second time.) Hence p divides both a and b.

Answer (ii) $(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b) + 6abc$.

$(a+b+c)(a^2+b^2+c^2) = a^3+b^3+c^3+(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)$.

From the second identity we can conclude that $p|(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)$; hence, from the first identity we get that $p|6abc$. Since p is a prime greater than 3, $p\nmid 6$. Hence $p|abc$. Thus p divides at least one of a, b and c. If $p|c$, then, since $p|(a+b+c)$, $p|(a+b)$ and, since $p|(a^2+b^2+c^2)$, $p|(a^2+b^2)$. By 2(i), p divides both a and b. Hence p divides each of a, b and c.

Cont.

Candidates found this the easiest question on the paper and there were many correct answers. (Note that an easier form was set for the Junior Examination.) Some people mentioned a generalization: If p is a prime greater than n, and $a_1, a_2, \ldots, a_n$ are integers such that p divides each of $\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} a_i^2, \ldots, \sum_{i=1}^{n} a_i^n$ then p divides each of $a_1, a_2, \ldots, a_n$.

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3 Consider the set of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

in which a_{11}, a_{22}, a_{33} are positive, all other coefficients are negative, and the sum of the coefficients in each equation is positive. Prove that the only solution of these equations is $x_1 = x_2 = x_3 = 0$.

Answer First of all, it is obvious that $x_1 = x_2 = x_3 = 0$ is a solution. It is the only solution if the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

Cont.

Adding columns 2 and 3 to column 1 we get

$$|A| = \begin{vmatrix} a_{11}+a_{12}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{22}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{32}+a_{33} & a_{32} & a_{33} \end{vmatrix}$$

where now each element of the first column is positive. Expanding by the first column,

$$|A| = (a_{11}+a_{12}+a_{13})(a_{22}a_{33}-a_{32}a_{23}) - (a_{21}+a_{22}+a_{23})(a_{12}a_{33}-a_{32}a_{13}) + (a_{31}+a_{32}+a_{33})(a_{12}a_{23}-a_{22}a_{13}).$$

As $(a_{22}+a_{21}+a_{23}) > 0$, $a_{21} < 0$ and $a_{23} < 0$, we have $a_{22} > |a_{23}|$ and similarly $a_{33} > |a_{32}|$; hence $a_{22}a_{33} > a_{23}a_{32}$ and so $(a_{11}+a_{12}+a_{13})(a_{22}a_{33}-a_{32}a_{23}) > 0$. It is easy to show that the two remaining terms in the expansion are positive by considering the signs of each component inside the second pair of brackets in each case. Thus $|A| > 0$ and so $|A| \neq 0$.

This question was poorly done, in general, with many getting no farther than saying $|A| \neq 0$ (or its equivalent, if the determinant notation was unknown). Very few mentioned the generalization to n equations in n unknowns.

One of the successful solvers used a completely different approach: Consider all possible ways in which the three elements of the solution triple, x_1, x_2, x_3, can have values that are positive, negative or zero. (Two examples: One is (-) and two are (+); one is 0, one is (+) and one is (-).) Then eliminate all cases except the one where all are 0. (The only difficult case is where all are (+), or, equivalently, all are (-). Can you deal with this case?)

4 (i) Prove that no 3 diagonals of a regular heptagon (7-sided polygon) are concurrent at a point other than a vertex of the heptagon. A diagonal is a line connecting two non-adjacent vertices.

4 (ii) How many points of intersection of pairs of diagonals lie within the heptagon?

4 (iii) Into how many compartments is the heptagon dissected by the diagonals?

4 (iv) Assuming that for n odd no 3 diagonals of a regular n-gon are concurrent, generalize (ii) and (iii) for the regular n-gon.

Answer (i) We would probably draw a sketch to start our thinking on the problem. (A sketch is not reproduced here.)
A diagonal joining one vertex to another next but one round the heptagon cannot be involved: Consider V_1V_3, say, where the vertices are named V_1, V_2, ..., V_7 in order round the heptagon. The only diagonals crossing it inside the heptagon are those from V_2 and no two of these can intersect both at V_2 and on V_1V_3. All told, there are 21 points of internal intersection on such diagonals (4 on any one.)
Suppose 3 of the remaining diagonals are concurrent at a point other than a vertex - 6 vertices will be involved in producing these diagonals. (Note that we cannot have 4 diagonals so concurrent as there are not 8 vertices to produce them.) This point represents the coalescence of the 3 possible points of intersections of pairs of the 3 diagonals. By symmetry, there must be 7 such points of concurrence, representing 21 possible points. But this would give 21+21=42 possible points, when we know from 4(ii) that there are only 35. Hence our supposition is wrong and the problem is solved.

Cont.

4 (iv) part (ii) First solution We set up a "counting process" to find P, the number of points of internal intersection. Consider all of the (n - 3) diagonals from V_1 in turn. V_1V_3 is crossed inside the n-gon only by diagonals from V_2 and so there are (n - 3) points on V_1V_3. V_1V_4 is crossed only by diagonals from V_2 and V_3 but one from each of the points (namely V_2V_4 and V_3V_1) intersects at a vertex; thus there are 2(n - 4) points on V_1V_4. Continuing in this way, there are 3(n - 5) points on V_1V_5, ... , (n - 2)(2) points on V_1V_{n-2}, and (n - 3)(1) points on V_1V_{n-1}. (Are you sure you can give the argument for V_1V_5?) The sum of these is

$$S = (n-3)+2(n-4)+3(n-5)+ \ldots +(n-3)(1)$$
$$= (n-2-1)+2(n-2-2)+3(n-2-3)+\ldots+(n-3)(n-2-(n-3))$$
$$= (n-2)(1+2+3+\ldots+(n-3))-(1^2+2^2+3^2+\ldots+(n-3)^2).$$

We look up a text-book to find that

$$(1+2+\ldots+N) = N(N+1)/2 \text{ and}$$
$$(1^2+2^2+\ldots+N^2) = N(N+1)(2N+1)/6.$$

(See, for example, Courant and Robbins: "What is Mathematics?" pages 12 and 14).

$$\therefore S = (n-2)(n-3)(n-2)/2 - (n-3)(n-2)(2n-5)/6$$
$$= (n-2)(n-3)\{3(n-2)-(2n-5)\}/6$$
$$= (n-1)(n-2)(n-3)/6.$$

S is the number of points on diagonals from V_1.

For the n vertices, it seems we would have nS points, but then each point would be counted 4 times (each point lies on 2 diagonals and each diagonal is considered twice - once from each end.) Hence $P = nS/4 = n(n-1)(n-2)(n-3)/24$.

This is a surprisingly neat answer after these long computations and, as always in such a case, we wonder if there is a neater way to obtain it.

Cont.

Second solution Any internal intersection is connected with 4 vertices, the ends of the 2 diagonals involved. It is easily seen that their 6 mutual joins will produce exactly one intersection inside the n-gon. Hence there are $\binom{n}{4}$ points.

This is a good example of a problem which can be solved in a straightforward, tedious way but which can be solved much more elegantly on adopting a different approach. It often happens that a more general approach and/or taking a "global view" of the situation leads to elegant solutions. In 4 (i), for example, use of the symmetry of the problem leads more quickly to a solution than does chasing around a few angles in search of a contradiction.

Answer 4 (iv) part (iii) A textbook gives Euler's formula specialized to a plane: "vertices" - "edges" + "faces" = 1 (Courant and Robbins, page 239). Now "vertices" = P+n (the n for the vertices of the n-gon. The number of "edges" on any one diagonal equals the number of points on it plus one. Adding, we get (2P+D) on all the D diagonals. As there are n sides to an n-gon,

"edges" = 2P+D+n

∴ "faces" = (2P+D+n) - (P+n) + 1 = P + D + 1.

Now D = $\binom{n}{2}$ - n since there are $\binom{n}{2}$ joins of the vertices of the n-gon, of which n are sides of the n-gon. (Another argument: There are (n-3) diagonals from any one vertex; considering the n vertices, D = n(n-3)/2, the half since each diagonal is counted twice.) Hence the number of compartments $= \binom{n}{4} + \binom{n}{2} - n + 1$. (This can be factorized to $(n-1)(n-2)(n^2-3n+12)/24$ which does not seem to be an especially interesting form of the result.)

Cont.

Answer 4 (ii) and (iii) We can read off the results, 35 and 50, respectively, from the general case above by putting n = 7. As an alternative, in desperation, we can count up on our initial sketch; most candidates did this and it's interesting to note that more decided on 49 compartments than on 50.

Question 4 was found to be the hardest. Although the first method for P was produced (but without the summations being done), nobody gave the second solution.

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5 Four equal weights are placed at different points on the edge of a uniform circular disc such that the centre of mass of the system is at the centre of the disc. Show that the weights must lie at the vertices of a rectangle.

Show by an example that it possible to arrange 6 equal weights on the edge of a uniform circular disc so that (i) the centre of mass is at the centre of the disc, and (ii) it is not possible to remove two weights without displacing the centre of mass.

Answer 5 (i) Clearly, by symmetry, the centre of mass of the disc itself is at its centre. Take 2 of the weights, m_1 and m_2 say, that are not on a diameter, and let the diameter which is the perpendicular bisector of their join be the x-axis. Let weight m_i be at (x_i, y_i) for i = 1,2,3,4. Thus $x_1 = x_2 = a$, say, and $y_1 = -y_2 = b$, say. So that $\sum_{i=1}^{4} m_i y_i = 0$ (as given $m_1 = m_2 = m_3 = m_4$) we must have $y_3 + y_4 = 0$.

Cont.

Hence, as the weights are on a circle, $x_3 = x_4$. So that $\sum_{i=1}^{4} m_i x_i = 0$, we must have $x_3 = x_4 = -a$.

Hence the weights are at (a,b), (a,-b), (-a,b), (-a,-b) and so lie on a rectangle.

Answer 5 (ii) There are infinitely many examples. One is to have a weight on the edge at either side of the x-axis at each of $x = -r/2$, $-r/4$, $3r/4$, where r is the radius of the disc. (Any such triplet of x-values adding to 0 will do, provided none is 0.)

A more symmetrical arrangement is obtained by putting 3 weights so that their radii from the centre are at $2\pi/3$ (the weights will be at the vertices of an equilateral triangle), and another 3 similarly arranged, displaced round the disc, but not so that the 6 form a regular hexagon. (We could also have a "no-displacement" case and so 2 weights at each of the 3 points, since this part of the question does not require the positions to be different.)

A number of people solved 5(ii) but few 5(i). Most approached part (i) by a building-up process: first putting on one weight and balancing it by a second diametrically opposite, and then placing a third anywhere to be balanced by a fourth diametrically opposite, whereas the question did not stipulate anything about where the centre of mass would be with only two weights.

Answer Obliterated Multiplication (see p. 28)

Seven hundred and seventy-five multiplied by thirty-three.