ANSWERS TO PROBLEMS IN PARABOLA VOL. 5, NO. 1

(Names of those sending correct solutions on p.12)

J111 The integer M consists of 100 threes (i.e. in decimal notation M = 3 3 3 ... 3 with 100 digits, all 3) and the integer N consists of 100 sixes. What digits occur in the product M.N.

Answer $M = \frac{1}{3}(10^{100} - 1)$ and $N = \frac{2}{3}(10^{100} - 1)$.

$$\text{Hence } M.N = \frac{2}{9}[10^{100} - 1]^2$$

$$= \frac{2}{9}[10^{200} - 2.10^{100} + 1]$$

$$= \frac{2}{9}[(10^{200} - 1) - 2(10^{100} - 1)]$$

$$= K - L$$

where $K = \frac{2}{9}(10^{200} - 1)$ is the integer consisting of 200 twos and $L = \frac{4}{9}(10^{100} - 1)$ is the integer consisting of 100 fours. Subtracting, we obtain 222 ... 21777 ... 78, in which there are 99 twos, a one, 99 sevens and an 8.

J112 In how many ways can 2^n be expressed as the sum of four squares of integers.

Answer If $n = 0$, $2^n = 2^0 = 1$ which is expressible in only one way, as $1^2 + 0^2 + 0^2 + 0^2$ (except for changes of order).

Cont.

If $n = 1$, $2^n = 2^1 = 2$ which is expressible in only one way as $1^2 + 1^2 + 0^2 + 0^2$.

If $n = 2$, $2^n = 2^2 = 4$ which is expressible in two ways as $2^2 + 0 + 0 + 0$, or as $1^2 + 1^2 + 1^2 + 1^2$.

If $n \geq 3$, 2^n is divisible by 8. Now the square of an odd number leaves a remainder of 1 on division by 8. [$(2k + 1)^2 = 8.\frac{k(k+1)}{2} + 1$ where $\frac{k(k+1)}{2}$ is an integer, as either k or k + 1 must be even.] It follows that $x_1^2 + x_2^2 + x_3^2 + x_4^2$ cannot be divisible by 8 if any of the x_i are odd numbers. Therefore,

$$\text{if } n \geq 3 \text{ and } x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^n \quad (1)$$

every x_i is even, $x_i = 2X_i$ say, and

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 2^{n-2}. \quad (2)$$

Every solution of (1) may be obtained by doubling up the X_is in a solution of (2), so that (1) and (2) have the same number of solutions. Hence, repeating the argument, the number of solutions of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^n$$

is the same as the number of solutions of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^1$$

if n is odd, namely 1, or of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^2$$

if n is even, namely 2.

0113 (a) If x, y and z are integers such that
$x^2 + y^2 + z^2 = 2xyz$, show that
$x = y = z = 0$.

(b) Find all integer solutions of
$w^2 + x^2 + y^2 + z^2 = 2wxyz$.

Answer (a) Assume, on the contrary, that there exist positive integers satisfying the equation. If p is a common factor of x and y then p^2 divides every term in the R.H.S. of $z^2 = 2xyz - x^2 - y^2$ and hence p also divides z. The factor p^2 can then be cancelled out to obtain

$$X^2 + Y^2 + Z^2 = 2pXYZ$$

where $x = pX$, $y = pY$, $z = pZ$. Further common factors may be cancelled out in the same way so that we eventually obtain an equation of the type

$$x_1^2 + y_1^2 + z_1^2 = 2kx_1y_1z_1 \quad (1)$$

in which x_1, y_1, and z_1 are relatively prime inpairs (i.e. no two have any common factor).

There is therefore at most one even number amongst x_1, y_1 and z_1. Since the R.H.S. of (1) is even they are not all odd; hence there must be exactly one even number amongst them. Without loss of generality we may suppose that $z_1 = 2Z_1$, the other two x_1 and y_1 being odd.

Cont.

But even this is impossible, since the L.H.S. of (1) leaves a remainder of 2 on division by 4 *, whilst the R.H.S. = $4kx_1y_1z_1$ is divisible by 4. Hence there are no non-zero solutions.

<u>Answer (b)</u> The same sort of argument shows again that the only solution is

$w = x = y = z = 0.$

Suppose that $w^2+x^2+y^2+z^2 = 2wxyz \neq 0$. Since the square of an odd number leaves a remainder of 1 on division by 8, $w^2+x^2+y^2+z^2 \equiv n \pmod 8$ where n is the number of odd terms on the L.H.S. But for n equal to 1, 2, 3 and 4, the R.H.S., $2wxyz \equiv 0, 0, 0$ or 4, and 2 or 6 (mod 8) respectively. But if all variables are even, (n=0) similar contradictions are obtained after the largest possible power of 2 is cancelled out. Hence the only solution is $w = x = y = z = 0$.

<u>0115</u> Find all solutions in integers of

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

<u>Answer</u> Rearrange to obtain $x = \frac{yz}{y-z}$ (1).
Let the h.c.f. of y and z equal d, so that $y = dY$, $z = dZ$ where Y and Z are relatively prime. Substitution in (1) gives

$$x = d.\frac{YZ}{Y-Z} \qquad (2)$$

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* Each of the odd squares leaves a remainder of 1 on division by 4.

Cont.

The integer Y - Z is relatively prime to both Y and Z (since otherwise these two numbers would not be relatively prime) and therefore it must divide d exactly. Put Y - Z = a and d = a.b in (2) to obtain

$$x = b(Z + a)Z$$
$$y = dY = ab(Z + a) \quad (3)$$
$$z = dZ = abZ$$

The three equations (3) in which a, b and Z are any three integers ($\neq 0$) with a and Z relatively prime give the general solution of the problem.

0116 By adding parentheses a^{b^c} can mean one of two different things viz. either $a^{(b^c)}$ or $(a^b)^c = a^{bc}$. How many different meanings can be given to $a^{b^{c^{d^e}}}$ by the addition of parentheses? How many different meanings can be given to

$$a^{b^{c^{\cdot^{\cdot^{\cdot^{k}}}}}}$$

if there are n symbols, all different?

Answer For ease of writing we shall change our notation, writing xPy to mean x^y. Let C_n be the number of different expressions obtainable by parenthesizing $a_1Pa_2Pa_3P \ldots Pa_n$ (1). Thus $C_1 = C_2 = 1$, and $C_3 = 2$ (since $a_1Pa_2Pa_3$ is either $a_1P(a_2Pa_3)$ or $((a_1Pa_2)Pa_3)$.

Cont.

When (1) is parenthesized, let the last operation of raising to a power be that between a_r and a_{r+1} i.e.

$$(a_1 P a_2 P \ldots P a_r) P (a_{r+1} P a_{r+2} P \ldots P a_n)$$

where the expressions in brackets are further parenthesized. There are C_r ways of parenthesizing the first expression and C_{n-r} ways of parenthesizing the second, so that altogether $C_r . C_{n-r}$ different expressions can be so obtained. Hence, summing for all positions of the last operation,

$$C_n = C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_r C_{n-r} + \ldots + C_{n-1} C_1$$

$$= \sum_{r=1}^{n-1} C_r C_{n-r} . \quad (2)$$

This is a "recursion formula" enabling C_n to be calculated if $C_1, C_2, \ldots C_{n-1}$ are already known. For example,

$$C_4 = C_1 C_3 + C_2 C_2 + C_3 C_1 = 1.2 + 1.1 + 2.1 = 5$$

and

$$C_5 = C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 = 14.$$

One fairly sophisticated method of obtaining a direct formula for C_n from (2) makes use of a "generating function", $f(t)$ defined by

$$f(t) = C_1 + C_2 t + C_3 t^2 + \ldots + C_n t^{n-1} + \ldots \quad (3)$$

Cont.

If this infinite series is squared, one obtains $[f(t)]^2 =$

$$C_1^{\,2} + (C_1C_2 + C_2C_1)t + (C_1C_3 + C_2C_2 + C_3C_1)t^2 + \ldots$$
$$(C_1C_{n-1} + C_2C_{n-2} + \ldots + C_{n-1}C_2)t^{n-2} + \ldots$$
$$= C_2 + C_3t + C_4t^2 + \ldots + C_nt^{n-2} + \ldots \quad \text{(using (2))}$$
$$= \frac{f(t)-C_1}{t} = \frac{f(t)-1}{t}.$$

Hence $f(t)$ is a root of the equation

$$tf^2 - f + 1 = 0$$

i.e. $f(t) = \dfrac{1 \pm \sqrt{1-4t}}{2t}$

Since $f(t) = 1$ when $t = 0$, we must take the root with the negative sign

$$f(t) = \frac{1-(1-4t)^{1/2}}{2t}.$$

Expanding by the binomial expansion theorem

$$f(t) = \frac{1}{2t}[1 - (1 - \frac{1}{2}.4t + \frac{\frac{1}{2} - \frac{1}{2}}{1.2}(-4t)^2 + \ldots$$
$$+ \frac{\frac{1}{2}.-\frac{1}{2}.-\frac{3}{2}\ldots-\frac{2k-3}{2}}{k!}(-4t)^k \ldots)].$$

Since, from (3), the coefficient of t^{k-1} is C_k we have $C_k =$

$$\frac{1}{2}.\frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}\ldots\frac{2k-3}{2}}{k!}.4^k = \frac{1}{k+1}\,\frac{1.3.5.\ldots(2k-3)}{k!}\,2^{2k}$$
$$= 2^{k-1}\frac{1.2.3.4.(2k-3)(2k-2)}{k!\;2.4.\ldots 2k-2}$$
$$= \frac{2^{k-1}(2k-2)!}{k!2^{k-1}(k-1)!} = \frac{1\;(2k-1)!}{(2k-1)k!\;(k-1)!}$$
$$= \frac{1}{2k-1}\;{}^{2k-1}C_{k-1}.$$

0117 At a certain reunion, each man present shook hands with an odd number of women and each woman present shook hands with an odd number of men. Prove that there were an even number of adults present.

Answer If $x_1, x_2, x_3, \ldots, x_n$ are the odd numbers of handshakes made by each of the n people present then $x_1 + x_2 + x_3 = \ldots + x_n$ is twice the total number of handshakes (since each handshake has been counted twice.) Since this total is even, n must be even, as the sum of an odd number of odd numbers is odd. [In the above, handshake means "handshake between a man and a woman". Handshakes between people of the same sex are ignored.]

0118 (Philip Hall's marriage problem).

The men in college X wish to get married to some of the girls in college Y. Each man x in college X picks out a set x' of girls in college Y, any of whom is acceptable to him; but he refuses to marry any girl not in the set x'. Under what conditions is it possible for all the men to marry? First, it is necessary that no x' be empty; but that is not enough - it could be that each man picks only one girl, and that they all pick the same girl. It is clearly necessary that, for every 2 men x and y, $x' \cup y'$ should contain at least 2 girls. Again, this is not sufficient. It is clearly necessary that, for every subset A of X, the set

$$A' = \bigcup_{x \in A} x'$$

have at least as many elements as A. Prove that this condition is sufficient.

<u>Answer</u> We use mathematical induction on n, the number of men in college X. If $n = 1$, the sufficiency of the condition is obvious. Assume that it is sufficient for all values of n less than k, and that there are k men in college X. We distinguish two cases as follows:-

Case (a): For every subset of men, A, which does not contain all k men, the corresponding set of women, $A' = \bigcup_{x \in A} x'$ (i.e. the set of all women acceptable to at least one man in A) is more numerous than A.

Case (b): On the contrary, there exists some set of r (< k) men, A, such that $A' = \bigcup_{x \in A} x'$ contains exactly r women.

<u>Case (a)</u> Let any man, x_1, marry a woman, y_1, who is acceptable to him. For any other man x, let x" represent his list of acceptable girls with y_1 deleted (if it was ever on it). If B is any set of the remaining (k - 1) unwed men, then $B'' = \bigcup_{x \in B} x''$ is either equal to $B' = \bigcup_{x \in B} x'$ or is equal to B' minus the woman y_1. (The first possibility occurs if no man in B regarded y_1 as acceptable.) In either case, since B' was more numerous than B, B" has at least as many elements as B. Accordingly, by our induction assumption the (k - 1) as yet unwed men can be paired with acceptable women.

Cont.

Case (b) By our induction assumption, the r men in the set A can be married to the r women in the set A'. For any other man x, let x" represent his list of acceptable women with any members of A' deleted. If B is any set s of the remaining (k - r) men, let $B'' = \bigcup_{x \in B} x''$ contain t women. Since $A \cup B$ is a set of r + s men $(A \cup B)'$ is a set of at least (r + s) women. But $(A \cup B)' = A' \cup B''$, a set of (r + t) women. Hence $t \geq s$ for any choice of B.
Hence, again using our induction hypothesis the unwed (k - r) men can be matched with acceptable women.
Hence, if the condition is sufficient for fewer than k men it is sufficient for k men, and the proof is complete.

0119 On a desert island there are n men and n women. Each man lists the women in his order of preference, and each woman lists the men in her order of preference. Suppose they pair off and marry. This set of marriages will be called unstable if there are a man and a woman, not married to one another, each of whom prefers the other to his or her present spouse; for then these two are apt to run off together. Prove that it is possible to avoid such trouble - a stable set of marriages always exists. (Note that it does no harm if a man prefers another woman to his wife, as long as this other woman prefers her own husband.)

Answer Let the men and women form two lines. The first man goes and stands next to the woman who heads his list. The second man then does likewise.

Cont.

If they are both standing with the same woman, she indicates which she prefers and the rejected man goes to the next woman on his list of preferences. This process is continued, the kth man visiting the women on his list in order, remaining with the first who prefers him to the man already with her, or who has not yet been approached (whichever occurs first). Similarly, any displaced man proceeds to the next woman on his list. (It is a waste of time his trying a woman higher up his list again as she is still standing either with the man she preferred to him previously, or with a man whom she rates even higher.) It is clear that the process comes to an end after a finite number of "moves", and if each man marries the woman with whom he is finally paired the set of marriages is stable, since there is no woman whom he prefers to his wife, who is not married to a man she prefers to him.

Obliterated Multiplication

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   X X X
     X X
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 X X X X
X X X X
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X X X X X
=========
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The puzzle is to replace each X by a prime digit not equal to one so that the multiplication is correct.

(Answer p. 44)

0120 Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$

It can be shown that there exist numbers e (≈2.17) and c (≈0.577) such that S_n becomes arbitrarily close to $\log_e n + c$ for all sufficiently large values of n.

Using this, show that if

$$t_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots - \frac{1}{2n}$$

$$= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$$

then t_{2n} becomes arbitrarily close to $\log_e 2$ for sufficiently large values of n.

Can you find the "limiting sum" of the infinite series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right) ?$$

Answer Note that $t_{2n} = (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n})$

$$- 2.(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n})$$

$$= S_{2n} - S_n$$

Given any h > 0, there exists an N such that

$$|S_n - (\log_e n + c)| < \frac{h}{2} \text{ for all } n > N.$$

Hence if n > N,

$$|t_{2n} - (\log_e 2n + c) + (\log_e n + c)|$$

$$= |[S_{2n} - (\log_e 2n + c)] - [S_n - (\log_e n + c)]|$$

$$\leq |S_{2n} - (\log_e 2n + c)| + |S_n - (\log_e n + c)|$$

$$< \frac{h}{2} + \frac{h}{2} = h.$$

Cont.

That is t_{2n} differs by an arbitrarily small amount, h, from $(\log_e 2n+c) - (\log_e n+c) = \log_e 2$ for sufficiently large values of n.

Again, setting $R_{3n} = \sum_{k=1}^{n} \left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right)$ it is simple to observe that

$$R_{3n} = S_{4n} - \tfrac{1}{2}S_{2n} - \tfrac{1}{2}S_n.$$

For a large enough value of n, the three terms on the R.H.S. can be made as close as desired to

$(\log_e 4n+c)$, $-\frac{1}{2}(\log_e 2n+c)$, and

$-\frac{1}{2}(\log_e n+c)$ respectively, and hence their sum R_{3n} can be made as close as desired to

$$\log_e \frac{4n}{\sqrt{2n}\ \sqrt{n}} = \log_e 2\sqrt{2} = \frac{3}{2}\log_e 2.$$

and this is the required limiting sum of the series.

[Since the nth term of the series becomes arbitrarily small for sufficiently large n, the "partial sums" R_{3n+1} will for large n differ negligibly from R_{3n} so that they also will be very close to $\frac{3}{2}\log 2$.]
