Parabola Volume 6, Issue 1 (1969)

The converse of a theorem on isosceles triangles

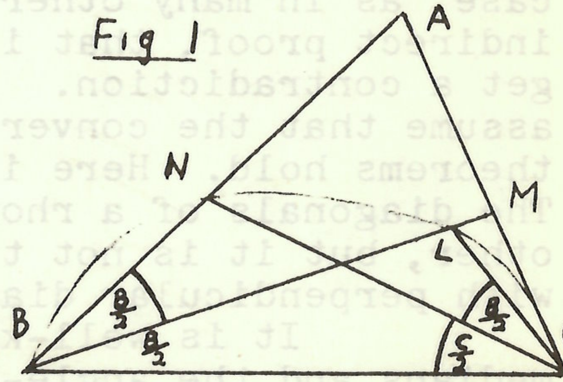
Converse theorems seem quite often too obvious and the beginner might think that they hold automatically. A well known example is the following theorem: The angles standing on the same arc in a circle are equal, and conversely, all points at which the same angle is subtended by an interval AB lie on a circular arc over AB. In this case, as in many others, we generally apply an indirect proof, that is, assuming the opposite, we get a contradiction. However, we should not simply assume that the converses of all mathematical theorems hold. Here is an easy counterexample: The diagonals of a rhombus are perpendicular to each other, but it is not true that all quadrilaterals with perpendicular diagonals are rhombuses.

It is well-known that the altitudes, the medians and the angle-bisectors belonging to equal sides of an isosceles triangle are of equal length. Conversely, if two altitudes of a triangle are of equal length, the corresponding two sides are also equal. This and the corresponding one about the medians is easily proved on the level of high-school mathematics. Surprisingly, the one about the angle-bisectors is quite difficult to prove. It was in 1840 that Lehmus suggested that if two angle-bisectors of a triangle are of equal length, the triangle is isosceles. The famous Swiss geometer, Jacob Steiner, was the first to prove it, therefore it is now known as the Steiner-Lehmus theorem. One expects a simple proof to a simple theorem like this one; Steiner's proof, however, was very involved. During the last hundred years there were more than 60 proofs published. A particularly simple proof is found in H.S.M. Coxeter's Introduction to Geometry. It appears there as an exercise, with a hint that enables the reader to finish the proof.

Cont.

Martin Gardner, reviewing the book for the Scientific American, commented upon the novel simplicity of this proof. This remark of Gardner's acted as a catalyst. He was flooded by letters of readers producing further proofs. One of these is quite remarkably elegant. It is due to G. Gilbert and D. MacDonnell and it was published in the Mathematical Monthly in 1963 (page 79). It is an indirect proof: Let us assume that in Figure 1, Triangle ABC is such that the lengths of the two angle bisectors BM and CN are equal but angle B is smaller than angle C. Then we can construct the line CL, as in Figure, making angle NCL equal $\frac{1}{2}$B, and L being a point on the bisector BM. Then the quadrilateral BNLC is concyclic, because the angles subtended by B and C upon the line segment NL are equal to each other, both being equal to $\frac{1}{2}$B.

Fig 1



Imagine the circle drawn around quadrilateral BNLC, and consider the two chords, CN and BL. According to our construction and the data, BL is shorter than CN. Therefore the acute angle subtended by any point on the circumference upon the chord BL has to be smaller than an acute angle standing on NC. The angle BCL = $\frac{1}{2}(\sphericalangle B + \sphericalangle C)$, is certainly an acute angle, because $\frac{1}{2}(\sphericalangle A + \sphericalangle B + \sphericalangle C) = 90^{o}$. Also the angle NBC = $\sphericalangle$B, is an acute angle, as it is not the largest angle in the triangle ABC. But we have assumed that $\sphericalangle$B is smaller than $\sphericalangle$C, therefore angle NBC = $\sphericalangle$B is smaller than $\sphericalangle$ LCB = $\frac{1}{2}(\sphericalangle B + \sphericalangle C)$. This is a contradiction, therefore the two angles B and C have to be equal.

Another proof was recently produced in Sydney, as an answer to Professor Butler's Brainteaser in the Sunday Telegraph.

Cont.

It is due to Dr B. Hagan and it competes in elegance and simplicity with the previous ones. Let us assume that in the triangle ABC of Figure 2 the two angle-bisectors BE and CD are of equal length, but angle C is greater than angle B. Construct DP and EQ, both perpendicular to BC. Then in the two right-angled triangles, CDP and BEQ, the two hypotenuses are equal, but angle EBQ (= ½ ∢B), is smaller than angle DCP (= ½ ∢C). Therefore EQ is smaller than DP. Now draw DX and EY both parallel to BC. As DP was found to be higher than EQ, DX has to be shorter than EY. Consider the triangle BEY. Angle YEB = angle EBC as alternating angles and therefore it is equal to angle YBE, because BE is an angle-bisector. This means that triangle BEY is isosceles. Similarly triangle DXC is isosceles. According to our data the bases of these two isosceles triangles are of equal length, but the base angles in triangle BEY are smaller than in triangle DCX. Then necessarily the length of side EY is shorter than that of DX in contradiction to our previous statement.

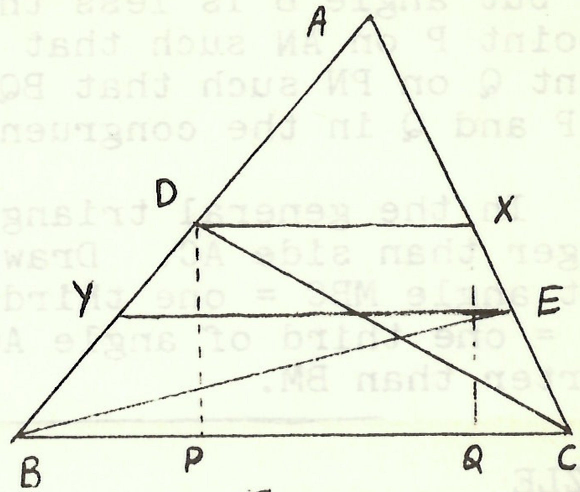


Fig 2

The remarkable history of this problem furnishes an example that it is still possible today to produce significant mathematical results with quite elementary means. To achieve greater simplicity is one aim of all scientific development.

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Suggested exercises:

1) Complete the proof found in Coxeter: Any triangle having two equal angle-bisectors is isosceles. (Hint: Assume as in Figure 1, that BM = CN, but angle B is less than angle C. Then there is a point P on AN such that angle PCN = $\frac{1}{2}\angle B$, and a point Q on PN such that BQ = CP. Compare the angles at P and Q in the congruent triangles BMQ and CNP.

2) In the general triangle ABC let side AB be longer than side AC. Draw the lines BM and CN such that angle MBC = one third of angle ABC and angle NCB = one third of angle ACB. Prove that CN is shorter than BM.

PUZZLE

Mr and Mrs Smith, Mr and Mrs Brown and Mr and Mrs Green, are seated equally spaced round a circular table. No man is sitting next to his wife, but each lady has a man on each side of her.
The names of the men and their wives, not necessarily respectively, are Tom, Dick and Harry, and Nancy, Joan and Mary.
The occupations of the men, again not necessarily respectively, are architect, politician and dustman.
Dick and Mr Smith often play bridge with the architect's wife and Mrs Green.
The dustman, who is an only child, has Mary on his right.
The politician is sitting nearer to Nancy than he is to Mrs Brown.
Harry is the architect's brother-in-law, and he has his only sister sitting on his left. The architect is sisterless.

Find their names, their occupations, and the order in which they are sitting round the table.

(Answer p.28)