IMPOSSIBLE CONSTRUCTIONS

It is fairly generally known, even amongst not very advanced students of mathematics, that in addition to the many ingenious constructions with straight edge and compasses which were discovered by the ancient Greeks, there were a number of similar construction problems which defied all their efforts, and the efforts of later generations of mathematicians for something like 2000 years, until it was eventually shown that these constructions were in fact impossible. Examples include the problem of trisecting a given angle, and the problem of duplicating a given cube (i.e. given a line segment AB it is required to construct a line segment CD such that the cube whose side is CD has exactly twice the volume of the cube whose side is AB). The nature of the impossibility proof is not nearly as generally known; it is the aim of this article to outline the ideas of the proof, although some of the algebraic details will have to be omitted.

In spite of the existence of this proof, from time to time people come forward claiming that they have discovered how to trisect angles. Examination of their attempts usually shows that they have not grasped the limitations implicit in the Greek notion of a construction. In particular it is imperative that the construction involve only a finite number of operations. The only allowable operations consist in drawing the straight line through two previously constructed (or initially given) points, and drawing the circle whose centre is a previously constructed point, and whose radius is the distance between two previously constructed points.

New points are "constructed" if they are points of intersection of two such straight lines and/or circles.

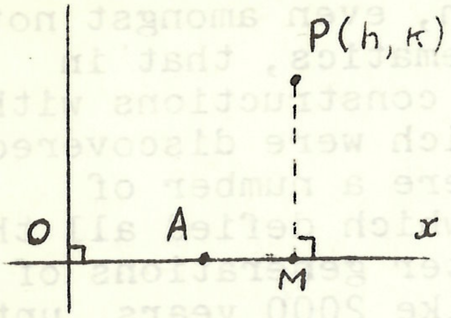


FIG. 1

A necessary ingredient of the discussion is the idea introduced by Descartes into geometry, namely, that it is useful to specify the position of a point by giving its "co-ordinates" relative to a pair of intersecting straight lines (axes).

In Fig. 1, the straight line OAMx is the "x-axis", and the straight line Oy at right angles to Ox is the "y-axis".

OA is a line segment of unit length. Given any point P, let PM be perpendicular to Ox. Then the x-co-ordinate of P is the number, h, of units of length in the "directed" line segment OM. (The "directed" here means that the co-ordinate has a plus or minus sign attached according as M lies to the right or left of O.) Similarly the "y-co-ordinate" of P is the number, k, of units of length in the directed line segment MP. (It is positive if P lies above the x-axis and negative if P is below the x-axis.) There is a (1-1) correspondence between points of the plane and ordered pairs of real numbers (h,k).

A number h, is called a constructible number if (after the unit interval OA is given) it is possible to construct a line segment of length h units. Clearly the constructible points in the plane are just those points whose co-ordinates (h,k) are both constructible numbers. We seek some characterisation which will enable us to distinguish which numbers are constructible.

As a first step in this search we observe that if h and k are constructible numbers, so are $h + k$, $h - k$, hk, $\frac{h}{k}$ $(k \neq 0)$, and $\sqrt{h}$ (Fig. 2, Fig. 3, Fig. 4 show simple constructions to obtain hk, $\frac{h}{k}$, and $\sqrt{h}$.)

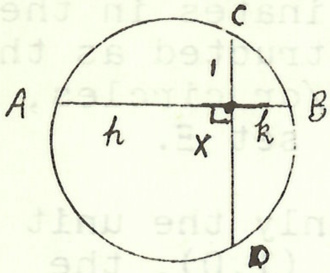


FIG. 2. DX* = hk

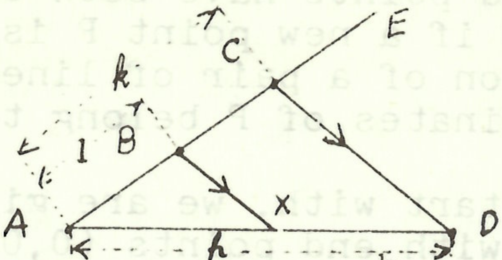


FIG. 3. AX* = $\frac{h}{k}$

Construct line AE, make AB* = 1, AC* = k. Join CD. Construct BX parallel to CD cutting AD at X.

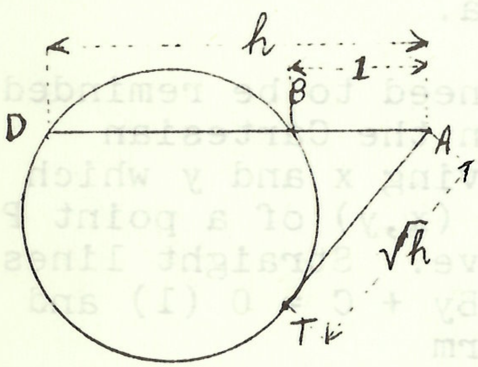


FIG. 4.

Let us denote by Ξ the set of all numbers obtainable from 1 by using any finite sequence of operations of addition, subtraction, multiplication, division or extraction of square roots. (For example, the number $\sqrt{\dfrac{\sqrt{2+4\sqrt{3}} + 5}{6 - \frac{7}{3}}}$ belongs to Ξ.) Then the above observation shows that all numbers in Ξ are constructible.

Conversely, it is not very difficult to show that <u>all</u> constructible numbers are in the set Ξ. To do <u>this</u> we establish the following lemma.

<u>Lemma</u> If after some sequence of constructions all constructed points have both co-ordinates in the set Ξ, and if a new point P is constructed as the intersection of a pair of lines and/or circles, then the co-ordinates of P belong to the set Ξ.

If, to start with, we are given only the unit interval, with end points (0,0) and (1,0), the condition of the lemma is obviously satisfied, and it is clear that we can never construct any point whose co-ordinates do not belong to Ξ. We proceed to outline the proof of the lemma.

Most of our readers will not need to be reminded that associated with any curve in the Cartesian plane there is an equation involving x and y which is satisfied by the co-ordinates (x,y) of a point P if and only if P lies on the curve. Straight lines have equations of the form $Ax + By + C = 0$ (1) and circles have equations of the form

$$x^2 + y^2 + 2Gx + 2Fy + D = 0. \qquad (2)$$

<u>Proposition 1</u> (a) Let ℓ be a straight line passing through the points (a,b) and (c,d), where all of a, b, c and d belong to the set Ξ. Then ℓ has an equation of form (1) in which A, B and C all belong to Ξ.

(b) Let k be a circle with centre (a,b) and radius r where a, b and r belong to b. Then k has an equation of the form (2) in which G, F and D all belong to b.

<u>Proofs</u> 1(a) The equation $(d-b)x + (a-c)y + (bc-ad) = 0$ is of form (1) and is therefore the equation of a straight line. It is obviously satisfied by $x = a$, $y = b$, and by $x = c$, $y = d$, so the line passes through these two points. The coefficients $A = (d-c)$, $B = (a-c)$, $C = (bc-ad)$ belong to Ξ.

1(b) (Fig. 5). By applying Pythagoras theorem to the triangle CMP, we see that P(x,y) lies on the circle if and only if $(x-a)^2 + (y-b)^2 = r^2$. This gives an equation of type (2) with $G = -a$, $F = -b$ and $D = a^2 + b^2 - r^2$ which all belong to Ξ.

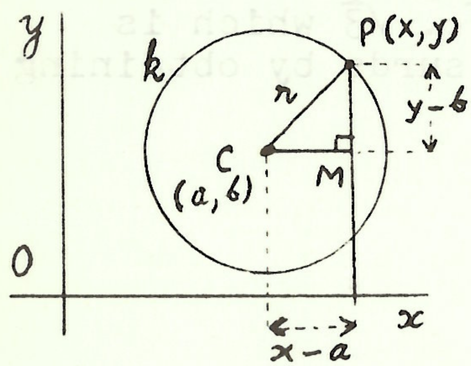


FIG. 5.

<u>Proposition 2</u> A point of intersection of two lines, a line and a circle, or two circles, having equations of forms (1) or (2) with all coefficients in Ξ, has co-ordinates belonging to Ξ.

<u>Proof</u> We illustrate by taking a line

$$Ax + By + C = 0 \quad (1)$$ and a circle

$$x^2 + y^2 + 2Gx + 2Fy + D = 0 \quad (2)$$ with all of A, B, C, D, F, G in b.

Solving these simultaneous equations for the co-ordinates (x,y) of the points of intersection yields

$$x = \frac{-AC-B^2G+ABF\pm\left((AC+B^2G-ABF)^2-(A^2+B^2)(C^2-2FBC+B^2D)\right)^{\frac{1}{2}}}{A^2+B^2}$$

and a similar expression for $y \left(= \frac{C-Ax}{B}\right)$, which are clearly in Ξ.

We can safely leave the similar treatment of the other two cases to our readers. It should also be clear that the lemma is a simple consequence of propositions 1 and 2.

The numbers in Ξ have another property which we now mention. Consider, for example, the number $x = \sqrt{2} + \sqrt{3} + \sqrt{6} = \sqrt{2} + \sqrt{3} + \sqrt{2}\,.\,\sqrt{3}$ which is certainly in Ξ. We can remove surds by obtaining in succession

$$\sqrt{2} = \frac{x-\sqrt{3}}{1+\sqrt{3}}$$

$$2 = \frac{(x^2-2\sqrt{3}x+3)}{1+2\sqrt{3}+3}$$

$$8-x^2-3 = -2x\sqrt{3}-4\sqrt{3}$$

$$\sqrt{3} = \frac{x^2-5}{2x+4}$$

$$3 = \frac{x^4-10x^2+25}{4x^2+16x+16}$$

$$x^4-22x^2-48x-23 = 0.$$

This shows that x is a root (or zero) of a polynomial $x^4-22x^2-48x-23$ whose coefficients are integers and whose degree is 4, a power of 2.

Investigation reveals that x does not satisfy any such polynomial equation of smaller degree (with integer coefficients). It is called the minimal polynomial satisfied by x, and x is called an algebraic number of degree 4. (Note that x satisfies many polynomials of any larger degree obtained simply by multiplying its minimal polynomial by an arbitrary polynomial. Note also that minimal polynomials can not be factorised into factors with integer coefficients. (Why?) Because of this they are called "irreducible polynomials".)

The following statement may now seem plausible.

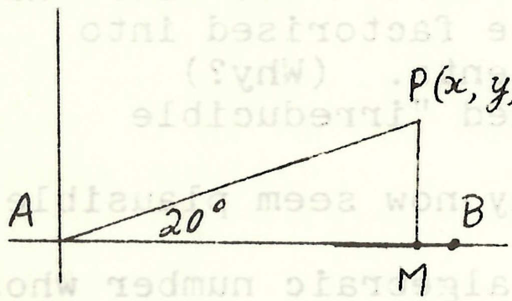
Th. Every number in Ξ is an algebraic number whose degree is a power of 2. That is each such number is a zero of an irreducible polynomial with integer coefficients whose degree is a power of 2.

We shall not prove this result. We draw the immediate inference:

Th. If x satisfies an irreducible polynomial equation with integer coefficients whose degree is not a power of 2, then x is not a constructible number.

As a particular example, consider $x = \sqrt[3]{2}$. The minimal equation satisfied by x is $x^3 - 2 = 0$ and it follows from the above theorem that $\sqrt[3]{2}$ is not constructible. This proves the impossibility of duplicating the cube, which requires the construction of a line segment whose length is $\sqrt[3]{2}$ times the given one.

What about trisecting angles? If there were a general method of doing this, since an angle of 60^o is very easily constructed we could also construct an angle of 20^o, /BAP, where AB is the given unit interval and AP is also of unit length.



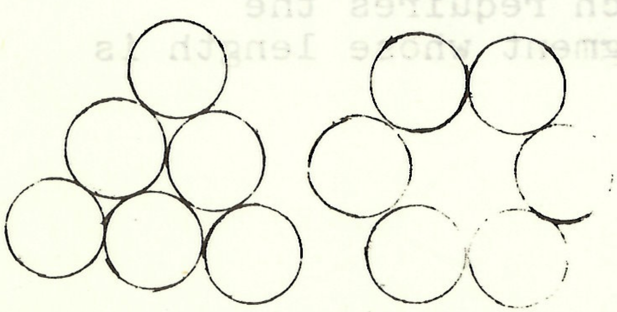
By elementary trigonometry, the x-co-ordinates of P would then be $x = \cos 20^o$. In the trigonometrical identity $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$ put θ equal to 20^o to obtain

$$\frac{1}{2} = 4x^3 - 3x$$

$$8x^3 - 6x - 1 = 0.$$

It is not difficult to prove that the polynomial on the left is irreducible, and it then again follows from the theorem above that x is not constructible. Hence there cannot exist any construction for trisecting angles.

Pennies Six pennies are arranged in a triangle as shown. The puzzle is to move them into the circular formation in the smallest number of moves. Each move consists in sliding one penny, without disturbing any other, to a new position in which it touches two others.



(Answer p. 28)