ANSWERS TO PROBLEMS IN PARABOLA VOL.5, NO. 2

(Names of those sending correct solutions on p.

J121 The numbers a,b,c,d and e are consecutive integers, each smaller than 10,000. Determine these numbers if $a + b + c + d + e = x$ is a perfect cube, and $b + c + d = y$ is a perfect square.

Answer $a + b + c + d + e = \quad = m^3$ (1)

$b + c + d \quad = \quad 3c = n^2$ (2)

From (1), 5 is a factor of m, hence 5^3 is a factor of m^3, and 5^2 is a factor of c.
From (2), 3 is a factor of n, hence 3^2 is a factor of n^2, and 3 is a factor of c. Returning to (1), 3 must be a factor of m, so that c must have a factor of 3^3. Thus c is a multiple of $5^2 \times 3^3 = 675$. If p is any further prime factor of c, it follows from (1) and (2) that p^6 divides c. Hence the only possible values of c are given by

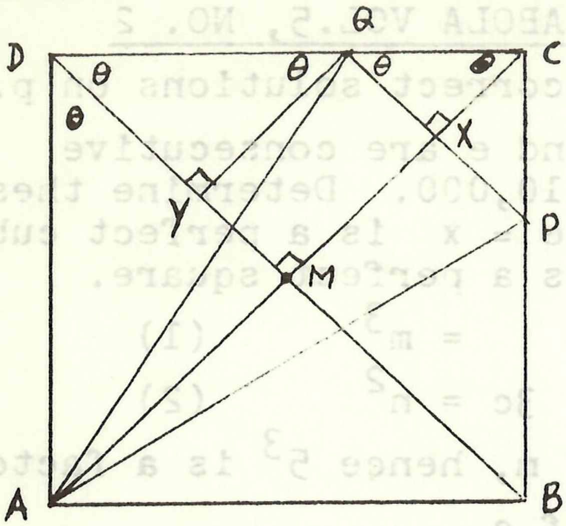
$c = 675 \times k^6$ where k is some integer. The only such number less than 10,000 is that obtained by taking $k = 1$, giving $c = 675$, the other integers being 673, 674, 675, 676 and 677.

J122 A square ABCD has sides of length a and diagonals of length d. A triangle APQ is drawn such that P is on the side BC, Q is on the side CD and AP = AQ. Let this triangle have perimeter p. Prove that

a) p is greater than 2d
b) p is less than 4a
c) p is less th in or equal to 2a + d.

Cont.

Answer In the diagram all the angles labelled θ are half right angles, whence $CX* = QX*$ and $DY* = QY*$.



$$(a)\quad \frac{p}{2} = AQ* + QX*$$
$$= AQ* + XC*$$
$$> AX* + XC*$$
$$= AC* = d.$$

$$(b)\quad \frac{p}{2} = AQ* + QX*$$
$$< (AD* + DQ*) + QC*$$
$$= AD* + DC* = 2a.$$

If Q does not coincide with D

$$(c)\quad \frac{p}{2} = AQ* + QX*$$
$$< (AY* + YQ*) + MY*$$
$$< AD* + DY* + YM*$$
$$= AD* + DM*$$
$$= a + \frac{d}{2}.$$

Equality occurs if Q coincides with D.

J123 n! (read n factorial) is defined to be the product of all the integers from 1 to n inclusive i.e. $n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n$.

a) which is the larger, $\sqrt[8]{8!}$ or $\sqrt[9]{9!}$?

b) simplify the sum $1(1!)+2(2!)+3(3!)+\ldots+n(n!)$.

Answer (a) $1.2.\ \ldots\ .8 < 9.9.\ \ldots\ .9$ (eight factors)

i.e. $8! < 9^8$.

Multiply by $(8!)^8$.

$$(8!)^9 < 9^8.(8!)^8 = (9!)^8.$$

Cont.

Take the 72nd root of both sides

$$(8!)^{9/72} < (9!)^{8/72}$$

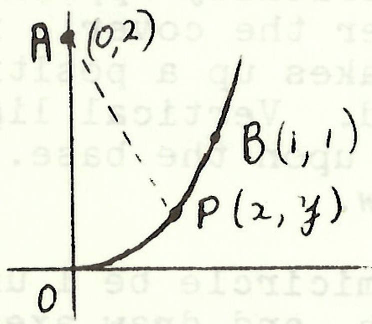
i.e. $$\sqrt[8]{8!} < \sqrt[9]{9!}.$$

(b) $$\begin{aligned}(n+1)! &= (n+1)(n!)\\ &= n(n!) + n!\\ &= n(n!) + (n-1)\{(n-1)!\} + (n-1)!\\ &= n(n!) + (n-1)(\overline{n-1}!) + (n-2)(\overline{n-2}!) + \\ &\quad \ldots + 2(2!) + 1(1!) + 1!.\end{aligned}$$

Hence $1.(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1.$

0124 Find the shortest distance from the point A(0,2) to that part of the parabola $y = x^2$ lying between $x = 0$ and $x = 1$.
You must prove that the distance found is the shortest.

Answer Let P(x,y) be a point on the arc OB of the parabola, $y = x^2$.



$$\begin{aligned}(AP^*)^2 &= x^2 + (2-y)^2\\ &= x^2 + (2-x^2)^2\\ &= x^4 - 3x^2 + 4\\ &= (x^2-\tfrac{3}{2})^2 + \tfrac{7}{4}.\end{aligned}$$

Since $0 \leq x^2 \leq 1$, it is clear that the minimum value is attained when $x = 1$, AP^* $(= AB^*) = \left((1-\frac{3}{2})^2 + \frac{7}{4}\right)^{\frac{1}{2}}$
$= \sqrt{2}.$

0125 Simplify $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

Answer Set $x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

In the identity $(A + B)^3 = A^3 + 3AB(A + B) + B^3$ put $A = \sqrt[3]{2+\sqrt{5}}$, $B = \sqrt[3]{2-\sqrt{5}}$ to obtain

$$x^3 = (2+\sqrt{5}) + 3(\sqrt[3]{2+\sqrt{5}}.\sqrt[3]{2-\sqrt{5}})(x) + (2-\sqrt{5})$$
$$= 4 + 3\sqrt[3]{(-1)}\,x$$
$$= 4 - 3x.$$

Hence $0 = x^3 + 3x - 4 = (x-1)(x^2+x+4)$. (1)

As x is a real number satisfying equation (1), we must have $x = 1$, since $x^2 + x + 4 = 0$ has no real roots.

0126 A container with the rectangular base ABCD has a transparent semi-cylindrical cover. A thin elastic thread joining the diametrically opposite vertices A and C is stretched over the cover - it may be assumed that the thread takes up a position such that its length is minimized. Vertical light rays case a shadow of the thread upon the base. Determine the shape of the shadow.

Answer Let the radius of the semicircle be 1 unit of length, and draw axes 0x, 0y parallel to the sides of the rectangle and intersecting at its centroid 0 (see Fig.1). In this figure APOC is the shadow and P', 0' are points on the thread vertically above P and 0. Let (x,y) be the coordinates of P with respect to the axes 0x, 0y.

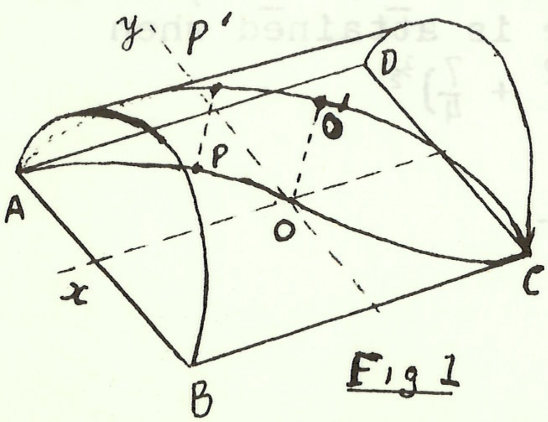
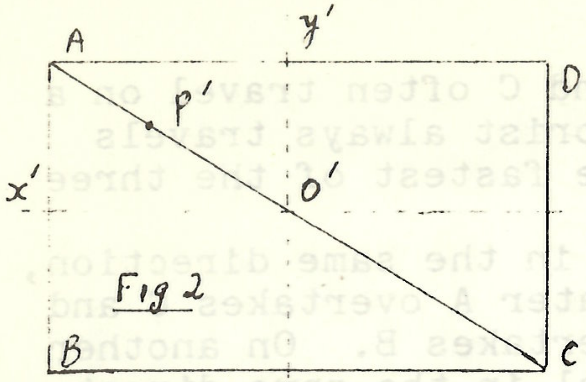
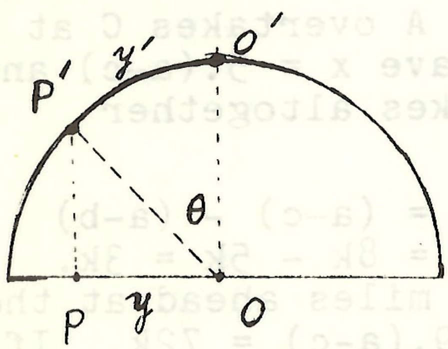


Fig 1

Cont.



The second figure shows the semi-cylindrical cover flattened out into a plane. Since the thread is in the position of minimum length it lies along the straight line joining A and C. The axes O'x', and O'y' lie vertically above Ox and Oy in Fig. 1. If (x',y') are the coordinates of P' with respect to O'x' and O'y's axes then the equation of the line AC is $y' = (\pi/\ell)x'$ where the length BC is ℓ units. Now it is obvious that the x-coordinates of P and P' are equal, $x = x'$.



The relation between the y-coordinates is easily seen from Fig. 3 (the end view) to be $y = \sin y'$. (Note that y' is the radian measure of the angle θ, since the radius of the semicircle is 1 unit). Substitution now gives $y = \sin \pi x'/\ell = \sin \pi x/\ell$ which is the equation of the curve made by the shadow. That is, the shape of the shadow is that of the portion of the sinusoidal curve $y = \sin \pi x/\ell$ with $-\ell/2 \leq x \leq \ell/2$.

0127 Three motorists A, B and C often travel on a certain highway, and each motorist always travels at a constant speed. A is the fastest of the three and C the slowest.
One day when the three travel in the same direction, B overtakes C, five minutes later A overtakes C and in another three minutes A overtakes B. On another occasion when they again travel in the same direction, A overtakes B first, then, nine minutes later, overtakes C. When will B overtake C?

Answer Let A's speed be a miles/minute.
" B's " " b " "
and " C's " " c " "

and on the first occasion let A be x miles behind at the instant B overtakes C. Since A overtakes C at a rate of a-c miles per minute we have $x = 5.(a-c)$ and similarly $x = 8.(a-b)$ since he takes altogether 8 minutes to gain x miles on B.

Hence $\frac{a-b}{5} = \frac{a-c}{8} = k$, say, and $b-c = (a-c) - (a-b)$
$= 8k - 5k = 3k$.

On the second occasion let C be y miles ahead at the instant A overtakes B. Then $y = 9.(a-c) = 72k$. If B takes t minutes to overtake C we have also $y = t.(b-c)$. i.e. $72k = t.3k$ whence $t = 24$ minutes.
i.e. B passes C 24 minutes after A passes B, or 15 minutes after A passes C.

0128 A pirate decided to bury treasure on an island near the shore of which were two similar boulders A and B and, further inland, three coconut trees C_1, C_2, C_3. Stationing himself at C_1, the pirate laid off C_1A_1 perpendicular and equal to C_1A and directed outwardly from the perimeter of triangle AC_1B.

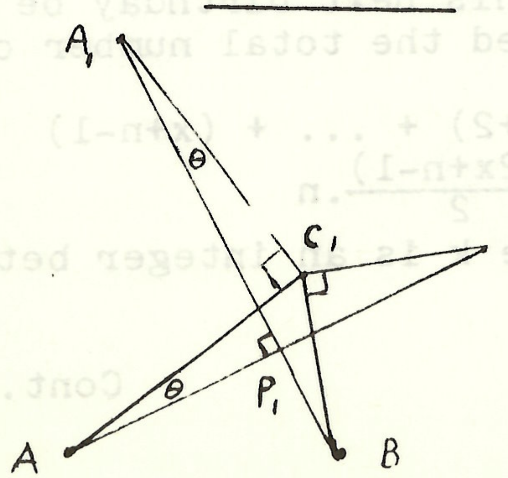
Cont.

He similarly laid off C_1B_1 perpendicular and equal to C_1B and also directed outwardly from the perimeter of triangle AC_1B. He then located P_1, the intersection of AB_1 and A_1B. Stationing himself at C_2 and C_3, he similarly located points P_2 and P_3, and finally buried his treasure at the circumcentre of triangle $P_1P_2P_3$. Returning to the island some years later, the pirate found that a storm had obliterated all the coconut trees on the island. How might he find his buried treasure?

<u>Answer</u> Since in the triangles AC_1B_1 and A_1C_1B we have $*AC_1 = *A_1C_1$, $*C_1B_1 = *C_1B$, and $A\hat{C}_1B_1 = A_1\hat{C}_1B$, the triangles are congruent. Hence $C_1\hat{A}_1B = C_1\hat{A}B_1 = \theta$, say, and $A\hat{P}_1B = P_1\hat{A}_1A + P_1\hat{A}A_1 = (45^o-\theta) + (45^o+\theta)$

$= 90^o.$

Since AB subtends a right angle at P_1, this point lies on the circle with diameter AB. Similarly P_2 and P_3 lie on this circle so that the circumcentre of the triangle $P_1P_2P_3$ is clearly the midpoint of the interval AB.



0129 A resident of Las Vegas, on his deathbed, held the following conversation with a friend who liked to solve problems.

"Over the last few years, I have saved every silver dollar that came into my hands. When I reached one hundred, I tied them in a bag. The first three hundred accumulated very fast and I hoped to reach a thousand, but I didn't make it. The several bags, each containing one hundred dollars, are in the closet of the next room. What I want you to do is to visit my son, who is a minor, on his next and each succeeding birthday and give him the number of dollars which equals his age. I figure that on your last trip you will have just used up all of the money."

"That's very interesting," said his friend. "Before I see the bags, let me try to figure out how many trips will be required." After an interval, the friend said, "I'll have to know your son's age." He was told. Then he said, "I now know how many trips will be required."

How many will he have to make?

Answer For the solution it is necessary to assume that he has filled actually more than three bags, though this is not made absolutely clear in the wording.

Let the son's age on his next birthday be x years. If n trips are required the total number of dollars handed over is

$$x + (x+1) + (x+2) + \dots + (x+n-1)$$
$$= \frac{(2x+n-1)}{2}.n$$

and this is 100k where k is an integer between 4 and 9 inclusive.

Cont.

Now $2x+n-1$ and n differ by $2x-1$, an odd number (not larger than 41 since $x \leq 21$) and it follows that they are of opposite parity. If, for example, there were four bags ($k = 4$), $(2x+n-1)n = 800$ and the only factorisation of 800 into one even factor and one odd factor differing by less than 41 is

$$800 = 32 \times 25, \text{ yielding}$$
$$2x + n - 1 = 32$$
$$n = 25, \text{ whence } x = 4.$$

$\therefore$ Listing the possibilities for 4, 5, 6 ... bags,

4 bags	$n = 25$,	$x = 4$
5 bags	$n = 25$,	$x = 8$
6 bags	$n = 25$,	$x = 12$
7 bags	$n = 25$,	$x = 16$
or	$n = 35$,	$x = 3$
8 bags	$n = 25$,	$x = 20$
9 bags	$n = 40$,	$x = 3$.

On being told the son's age, the friend was able to determine the number of trips required. Clearly he cannot have been informed that $x = 3$, since there would still be two possible values of n (35 and 40). Whichever other value of x he was given the only remaining value of n to occur is $n = 25$.

0130 "Did your teacher give you that problem?" I asked.
"It looks rather tedious."
"No," said Tim, "I made it up. It's a polynomial with my age as a root. That is, x stands for my age last birthday."
"Well, then," I remarked, "it shouldn't be so hard to work out - integer coefficients, an integer root. Suppose I try $x = 7$ . . . No, that gives 77."
"Do I only look 7 years old?" demanded Tim.
"Well, let me try a larger integer ... No, that gives 85, not zero."
"Oh, stop kidding!" said Tim looking over my shoulder. "You know I'm older than that!"
How old is Tim?

Answer First we observe (for use later) that if $P(y) = a_0 + a_1 y + \ldots + a_n y^n$ is a polynomial with integer coefficients and if h and k are any two integers, then the integer P(k) - P(h) is exactly divisible by the integer k - h. Indeed,

$$P(k) - P(h) = a_1(k-h) + a_2(k^2-h^2) + \ldots + a_n(k^n-h^n)$$
$$= (k-h)[a_1 + a_2(k+h) + \ldots$$
$$+ a_n(k^{n-1}+k^{n-2}h+ \ldots +kh^{n-2}+h^{n-1})]$$

and the factor enclosed by square brackets is clearly a whole number.

Let F(y) be Tim's polynomial, and let x be his age last birthday. Since x is a root of F(y), by the factor theorem F(y) = (y-x)P(y). Here P(y) is a polynomial with integer coefficients.

We are given that

$77 = F(7) = (7-x)P(7)$ (1) and that x exceeds 7 by more than 1. Since 7-x is a factor of 77 the possibilities are 7-x = -7, 7-x = -11 and 7-x = -77. The last of these, making Tim's age 84 years, could almost be discarded on commonsense grounds, but our readers may like to rule it out by the same sort of argument that we now use to eliminate the second possibility, x = 18. If x = 18, and n is the "larger integer" tried, we are given

$85 = (n-18)P(n)$ (2). Since $7 < n < 18$, (n-18) lies between -10 and -1 inclusive, and being a factor of 85 must either be -5 or -1, corresponding to n = 13 and n = 17 respectively.

Cont.

From (1) and (2) we would then have either

$$P(7) = -\frac{77}{11} = -7$$

and $P(13) = -17$

or $P(7) = -7$ and $P(17) = -85$.

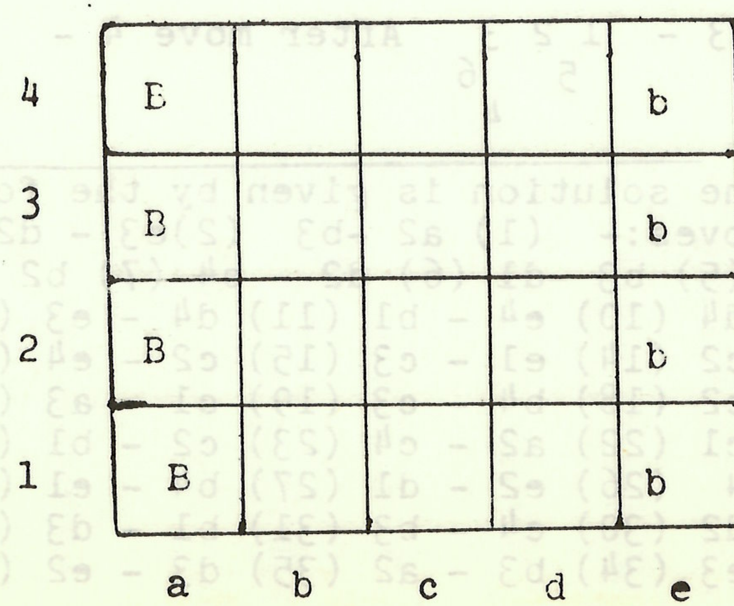
Each of these is ruled out by our initial observation, the first since $-17-(-7)$ is not exactly divisible by $13-7$, the second since $-85-(-7)$ is not divisible by $17-7$.

The only remaining possibility is $7-x = -7$, making Tim's age $x = 14$. (Tim's polynomial could have been $-3y^2 + 52y - 140$. The "larger integer" tried was 9.)

BISHOPS

In the accompanying diagram, the squares a1,a2,a3, and a4 are occupied by white bishops (the usual chess piece which moves any number of squares diagonally) and the squares e1, e2, e3, and e4 contain black bishops. By moving alternately white and black pieces, in 36 moves interchange the positions of the black and white pieces in such a way that no white bishop (B) is ever threatened by a black bishop (b).

(Answer p.28)



Answers

Amnesia (p.

Man's name	wife	occupation	statements uttered
ALF	ETHEL	D.S.	3, 12, 18
BERT	CLARISSA	D.K.P.	6,8,19
CHARLIE	FLOSSIE	B.W.	4,13,16
DOUG	BEATRICE	W.	5,7,20
ERNIE	GERTIE	W.O.	1,11,21
FRED	AGNES	D.O.	2,15,17
GEORGE	DIANA	S.U.	9,10,14.

Puzzle

Harry Brown, politician is seated diametrically opposite to his wife, Joan. On her right is the dustman, Dick Green, and on her left the architect, Tom Smith. Mary Smith and Nancy Green are opposite their husbands.

Pennies Label the pennies in the triangle thus -

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  1
 2 3
4 5 6
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One solution is given by the following diagrams.

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After move 1 -  1 2 3      After move 2 - 1 2 3
               4 5 6                        5 6
                                             4

After move 3 -  1 2 3      After move 4 -   2 3
                 5   6                     5   6
                   4                        1 4
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BISHOPS One solution is given by the following sequence of moves:- (1) a2 -b3 (2)e3 - d2 (3) a3 - b2 (4)e2 -d3 (5) b3 -d1 (6) d2 - b4 (7) b2 - c1 (8)d3 - c4 (9)a1 -d4 (10) e4 - b1 (11) d4 - e3 (12) b1 - a2 ((13)a4 - c2 (14) e1 - c3 (15) c2 - e4 (160 c3 - a1 (17) d1 - c2 (18) b4 - c3 (19) c1 - a3 (20) c4 - e2 (21) e3 - c1 (22) a2 - c4 (23) c2 - b1 (24) c3 - d4 (25)a3 - b4 (26) e2 - d1 (27) b4 - e1 (28) d1 - a4 (29) c1 - d2 (30) c4 - b3 (31) b1 - d3 (32 d4 - b2 (33) d2 - e3 (34) b3 - a2 (35) d3 - e2 (36) b2 - a3.