Parabola Volume 36, Issue 1 (2000)

THE MATHEMATICIAN ON THE BANKNOTE: CARL FRIEDRICH GAUSS

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PART ONE

Several years ago, a friend of mine in Germany sent me a German 10-mark banknote. It was unusual in that it featured the image and the work of a mathematician. This German mathematician was Carl Friedrich Gauss (1777–1855), whose work was so influential that he can lay claim to being the greatest mathematician of all time.

Gauss' discoveries encompass an astonishing variety of fields in pure mathematics, applied mathematics, physics, astronomy and geodesy. At the age of eighteen he discovered the construction of the regular 17-gon by ruler and compasses, something that had eluded the ancient Greeks. This result was included in his masterful book on Number Theory. He was the first to prove the Fundamental Theorem of Algebra, which states that every polynomial equation with real or complex coefficients has at least one root in the set of complex numbers. Gauss discovered non-Euclidean geometry, though he did not publish his work, but he wrote many papers on differential geometry. In physics he did important work on the Earth's magnetism and potential theory, and with Wilhelm Weber, built a primitive telegraph device for sending messages. The unit of magnetic induction is named after him. In 1807 he was appointed as Professor of Astronomy and Director of the Observatory at the university in Göttingen, a city in Germany, where he stayed until his death. He calculated the orbits of celestial bodies, and supervised the construction of the observatory. In geodesy (the study of the shape and size of the Earth), he made contributions to the theory and practice of measurement of the Earth's surface.

Given such a diversity of branches of science in which Gauss worked, it is interesting to see what the German government chose to illustrate on the banknote.

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The front of the banknote features a portrait of Gauss beside the Normal Curve, or, as it is sometimes called the Gaussian Error Curve. As we know, no measurement is exact, and any measurement of a quantity differs from the quantity's true magnitude. This difference is called the error. Gauss was interested in the distribution of the errors encountered when taking measurements.

Making the assumption that when any number of equally good direct measurements $M_1, M_2, M_3, \ldots,$ of an unknown magnitude X are given, the most probable value is their arithmetic mean, Gauss deduced that the errors were distributed *normally*, leading to the normal curve. His argument is outlined below.

Let us recall that, for a finite population, if a value x_k occurs with frequency f_k from n observations then its *relative frequency* p_k is given by $p_k = \frac{f_k}{n}$ and, for this population, we define the probability that x_k occurs to be its relative frequency. That is, we define the *probability distribution* of the associated *discrete* random variable X to be the *relative frequency distribution*, so that $P(X = x_k) = p_k$.

However, if the population consists of an interval of the whole real line, then the distribution is continuous, and we need to consider the probability of a value within a range of values, say a and b. For the continuous random variable X we say $P(a \leq X \leq b) = \int_a^b \phi(x)dx,$ where the function $\phi(x)$ is called the probability density function. The curve y $= \phi(x)$ takes the place of the relative frequency polygon in the discrete case.

A theorem of calculus (the Mean Value Theorem) states that, with $\phi(x)$ under certain conditions, there is a c between a and b where $\int_a^b \phi(x)dx = \phi(c)(b-a).$ Hence we have, for a continuous random variable, $P(a \leq X \leq b) = \phi(c)(b-a),$ where $a < c < b.$ Now if the interval $a < x < b$ is very small, then $\phi(a) \approx \phi(c),$ and so $P(a \leq X \leq b) \approx \phi(a)(b-a).$

In this way, Gauss supposed that the probability of an error lying in a small interval between Δ and $\Delta + d\Delta$ is $\phi(\Delta)(\Delta + d\Delta - \Delta) = \phi(\Delta)d\Delta.$ If ϵ is the least amount which the measuring instrument can measure, then it can be supposed that the possible values of any measurement increase by steps of ϵ, and the probability of an error between Δ and $\Delta + d\Delta$ may be taken as $\phi(\Delta)\epsilon$. Since this interval is small, we may say that the probability of the error *being* Δ is $\phi(\Delta)\epsilon$.

Now suppose that we have a quantity x whose true value is p, and we take s measurements $M_1, M_2, \ldots, M_s$ of x. The errors are $\Delta_1 = M_1 - p$, $\Delta_2 = M_2 - p$, $\ldots, \Delta_s = M_s - p$. The probability of the error in the first measurement being $M_1 - p$ is $\phi(M_1 - p)\epsilon$, the probability of the error in the second measurement being $M_2 - p$ is $\phi(M_2 - p)\epsilon$ and so on. If the measurements are all independent, then the probability that the measurements $M_1, M_2, \ldots, M_s$ will occur is therefore

$$\phi(M_1 - p)\phi(M_2 - p)\cdots\phi(M_s - p)\epsilon^s.$$

Although we will not do so here, it can be shown, using Bayes' Theorem in Inductive Probability, that the probability of the true value of x lying between p and $p + dp$ is

$$\frac{\phi(M_1 - p)\phi(M_2 - p)\cdots\phi(M_s - p)\,dp}{\int_{-\infty}^{\infty}\phi(M_1 - p)\phi(M_2 - p)\cdots\phi(M_s - p)\,dp},$$

and therefore the *most probable* estimate of p is that value of x which occurs when $\phi(M_1 - x)\phi(M_2 - x)\cdots\phi(M_s - x)$ is a maximum.

Hence we require
$$\frac{d}{dx}\big(\phi(M_1 - x)\phi(M_2 - x)\cdots\phi(M_s - x)\big) = 0$$

It is easier to differentiate here by taking logarithms (to base e) first.
Let $F(x) = \phi(M_1 - x)\phi(M_2 - x)\cdots\phi(M_s - x)$.
Then $\log F(x) = \log\phi(M_1 - x) + \log\phi(M_2 - x) + \cdots + \log\phi(M_s - x)$.
Differentiating both sides with respect to x we have

$$\frac{F'(x)}{F(x)} = \frac{d\log\phi(M_1 - x)}{dx} + \frac{d\log\phi(M_2 - x)}{dx} + \cdots + \frac{d\log\phi(M_s - x)}{dx}.$$

We want $F'(x) = 0$ and so we have

$$\frac{d\log\phi(M_1 - x)}{dx} + \frac{d\log\phi(M_2 - x)}{dx} + \cdots + \frac{d\log\phi(M_s - x)}{dx} = 0 \tag{0.1}$$

Here Gauss made his assumption that the most probable value of the measurements $M_1, M_2, \ldots, M_s$ is their arithmetic mean x. By definition

$$x = \frac{1}{s}(M_1 + M_2 + \cdots + M_s)$$
$$sx = M_1 + M_2 + \cdots + M_s$$

So
$$(M_1 - x) + (M_2 - x) + \cdots + (M_s - x) = 0. \tag{0.2}$$

Now both equations (0.1) and (0.2) really define the (same) value of x we are seeking, and hence they are equivalent. So each each term involving $M_i - x$ in (0.1) is proportional to the corresponding term in (0.2), and we have

$$\frac{d\log\phi(M_i - x)}{dx} = c(M_i - x) \quad \text{where } c \text{ is a constant.}$$

On integrating we have

$$\log \phi(M_i - x) = -\frac{c}{2}(M_i - x)^2 + k, \text{ where } k \text{ is a constant.}$$

Taking the exponential of both sides gives

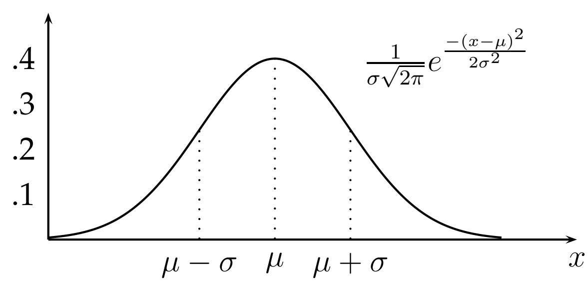
$$\phi(M_i - x) = Ae^{-\frac{1}{2}c(M_i - x)^2} \quad \text{where } A = e^k \text{ is a constant.}$$

So each error $M_i - x$ satisfies the equation $\phi(\Delta) = AE^{-\frac{1}{2}c\Delta^2}$.
Now the sum of the probabilities of all possible errors is 1 and so we have

$$1 = \int_{-\infty}^{\infty} \phi(\Delta)d\Delta = \int_{-\infty}^{\infty} Ae^{-\frac{1}{2}c\Delta^2}d\Delta.$$

But there is a standard result which says that $\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$, and so we can show $A = \sqrt{(c/2\pi)}$.
Writing h for $\sqrt{(c/2)}$, we have $\phi(\Delta) = \frac{he^{-h^2\Delta^2}}{\sqrt{\pi}}$.

This result shows that the distribution of the errors about the true value is what is called a normal frequency distribution with mean 0 and standard deviation $\frac{1}{h\sqrt{2}}$.
Gauss gave this proof in his book "Theory of the Motion of Celestial Bodies Moving around the Sun in Conic Sections", published in 1809.
The equation of the normal curve on the banknote is an extension of the above result which gives the normal distribution with mean μ and standard deviation σ:



The Normal Curve, as it appears on the front of the banknote

In the next issue of *Parabola* I will discuss the reverse side of the banknote.