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THE 72 RULE AND OTHER APPROXIMATE RULES OF COMPOUND INTEREST

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Introduction

There is a simple approximate rule of thumb used by investors and accountants to estimate the time taken in years, n, for an investment to double with an interest rate of $R\%$, or indeed for a debt to double if left unpaid. One simply divides 72 by R to estimate the time in years. For example an interest rate of 8% p.a. gives a doubling time of about 72/8 = 9 years. Alternatively we might ask what interest rate will cause a doubling in 10 years: answer 72/10 = 7.2%. We will see that these are very good approximations, but the rule doesn't work quite so well for interest rates very different from 8 or 9%. The rule is known as "Rule 72" or "the 72 Rule" and is often attributed to the investment advisor Henri Aram [1] who popularized it. A few years ago I spoke to Mr Aram after a lunchtime lecture he gave at the Sydney Stock Exchange and he told me he had first heard of it from an aged American investor while on an ocean cruise. The rule may also be found as an exercise in a respected textbook on mathematical investment analysis [2]. A practical virtue of the rule is the large number (10) of factors of 72, simplifying mental calculations. In subsequent sections reasons for the success of this and other similar rules are discussed.

Of course the time may be measured in periods other than years provided the interest rate is that for the corresponding period. In fact since the introduction of computers, lending authorities have tended to use shorter periods (eg. monthly) for reckoning accrued interest. In this context there is another approximation that is intrinsically interesting and sometimes convenient in calculations for long-term loans like mortgages from a bank.

Of course most pocket calculators nowadays will calculate compound interest without the need for approximations but simple rules give better intuitive information about the effects of long-term compound interest.

The 72 Rule

To derive the 72 rule we make use of a known series for the natural logarithm, $\ln x$ or $\log_e x$ based on the limiting sum of the geometric series

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \quad (-1 < x < 1).$$

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Integrating, $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{5} + \cdots$$
for $-1 < x \leq 1.$ For small values of x it is convenient to write,

$$\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$$

where $O(x^3)$ means terms in x^3 and higher, and that the approximation

$$\ln(1+x) = x - \frac{x^2}{2}$$

has an error of terms in x^3.

Now consider the investment of one dollar with compound interest $R\%$ payable yearly. After n years it will become $(1+\frac{R}{100})^n$ dollars or $(1+r)^n$ where $r = \frac{R}{100}$ is the fractional interest rate.

For this to represent a doubling in value we must have

$$\left(1 + \frac{R}{100}\right)^n \equiv (1+r)^n = 2. \tag{1}$$

Now take natural logarithms giving

$$n \ln(1+r) = \ln 2.$$

Then,

$$\begin{aligned}
n &= \frac{\ln 2}{\ln(1+r)} \\
&= \frac{\ln 2}{(r - \frac{1}{2}(r)^2 + O((r)^3))}, \qquad \text{substituting for } \ln(1+r) \\
&= \frac{\ln 2}{r(1 - \frac{1}{2}(r) + O((r)^2))} \\
&= \frac{\ln 2}{r}(1 + \frac{1}{2}(r) + O((r)^2)) \qquad \text{by long division for example} \\
&\approx \frac{\ln 2}{r}\left(1 + \frac{r}{2}\right) \qquad \text{neglecting the higher order terms.}
\end{aligned}$$

The main term on the right $\frac{\ln 2}{r}$ determines how n behaves for arbitrarily small interest rates. For practical interest rates, we might modify this

$$\frac{\ln 2}{r} \quad \text{to} \quad \frac{\ln 2}{r}\left(1 + \frac{r}{2}\right)$$

for some typical interest rate, and then we use the simpler approximation

$$n = \frac{K}{R} = \frac{K}{100r}$$
$$\text{with} \qquad K = 100 \ln 2 \left(1 + \frac{r}{2}\right) = 100 \ln 2 \left(1 + \frac{R}{200}\right).$$

Taking $R = 7\%$ to evaluate K gives $K = 71.74$, or $R = 8\%$ gives $K = 72.09$.
So for interest around 7 or 8%, $K = 72$ is pretty close. This is a justification for the 72 rule

$$n = \frac{72}{R} \qquad \text{or} \qquad R = \frac{72}{n}.$$

Alternatively we could treat it purely empirically by trying to find a functional form that represents the exact formula by some convenient numerical approximation. This might be regarded as an example of approximation, a branch of both mathematics and numerical computation. The theoretical development above suggests the type of approximation likely to be useful.

A more accurate approximation could be obtained from

$$n = \frac{100 \ln 2}{R}\left(1 + \frac{R}{200}\right) = \frac{100 \ln 2}{R} + \frac{1}{2}\ln 2$$

$$n - \frac{1}{2}\ln 2 = \frac{100 \ln 2}{R} \quad \text{or} \quad R = \frac{100 \ln 2}{n - \frac{1}{2}\ln 2}$$

or numerically

$$n - 0.3466 = \frac{69.31}{R} \qquad \text{or} \qquad R = \frac{69.31}{n - 0.3466}.$$

Although this is a good approximation we probably wouldn't choose to use it because it is hardly easier to use than the precise equations,

$$n = \frac{\ln 2}{\ln(1 + \frac{R}{100})}$$

or solving for R

$$1 + \frac{R}{100} = 2^{1/n}$$

$$R = 100(2^{1/n} - 1).$$

Numerical comparisons are given in Table 1 where the calculated interest rates for doubling over given periods are listed.

n Doubling time Years	R Interest Rates %		
	72 Rule	Exact Rule (3 dec)	Alternative rule (3 dec)
72	1	0.967	0.967
36	2	1.944	1.944
18	4	3.926	3.926
12	6	5.946	5.948
10	7.2	7.177	7.180
9	8	8.006	8.010
8	9	9.051	9.057
6	12	12.246	12.261
4	18	18.921	18.973
2	36	41.421	41.921

Table 1
Years and Interest rates for doubling

We see that the 72 rule gives fairly good results over a range of interest rates. The alternative rule gives better results over a bigger range but the price is a more complicated formula.

Table 1 gives the interest rate for doubling over a whole number of years. If the rules are used to find the number of years for a given interest rate the result might not be a whole number of years. For example the 72 rule at 10% p.a. gives 7.2 years. The original formula estimates the interest on the fractional part k $(0 < k < 1)$ of a year as

$$(1 + \frac{R}{100})^k,$$

whereas a financial institution would usually reckon instead with

$$\left(1 + \frac{kR}{100}\right).$$

This is another inherent approximation, not usually serious. In the example above with $k = 0.2$

$$\begin{aligned} (1 + \frac{R}{100})^k &= 1.1^{0.2} = 1.0192 \\ \text{and} \qquad 1 + \frac{kR}{100} &= \qquad 1.0200. \end{aligned}$$

Rules for other than doubling and their interrelationship

We have seen that the 72 rule represents doubling fairly accurately. Obviously if we estimate the time from the 72 rule and add that time again the original sum will quadruple, so we would have an 144 rule for quadrupling, i.e. $\frac{144}{R}$ gives the number of years to quadruple. Other constants than 144 or 72 will correspond to other multiples.

Now let's denote by $n(d)$ the number of years under compound interest with rate $R\%$ that leads to a multiplication by d. ($n(2)$ would be the n of the previous sections).

We have instead of equation (1)

$$\left(1 + \frac{R}{100}\right)^{n(d)} = d$$

from which

$$n(d) = \frac{\ln d}{\ln\left(1 + \frac{R}{100}\right)}$$

and following the previous approximation method for the 72 rule,

$$n(d) \approx \frac{100 \ln d}{R}\left(1 + \frac{R}{200}\right).$$

so

$$\begin{aligned} n(d) &\approx \frac{100\ln 2}{R}\left(1+\frac{R}{100}\right) \times \left(\frac{\ln d}{\ln 2}\right) \\ &\approx \frac{72}{R} \times \left(\frac{\ln d}{\ln 2}\right) \end{aligned}$$

using the approximation that gave the 72 rule.

We now write the rule as

$$n(d) = \frac{K(d)}{R}$$

where $K(d) = K(2) = 72$ for the doubling rule $d = 2$. Then,

$$K(d) = \left(\frac{72}{\ln 2}\right) \ln d.$$

For two values $d = a$ and $d = b$ we see that

$$K(ab) = K(a) + K(b)$$

because

$$\ln ab = \ln a + \ln b.$$

However, this relationship also follows from a direct application of the definitions of $n(a)$ and $n(b)$, since a time $n(a)$ multiplies the sum by "a" and a time $n(b)$ multiplies the sum by "b" so that time $(n(a)+n(b))$ multiplies by ab and,

$$n(ab) = n(a) + n(b)$$

irrespective of the approximation. Again,

$$n(a) = n\left(\frac{a}{b}\right) + n(b)$$

$$\text{or} \qquad n\left(\frac{a}{b}\right) = n(a) - n(b).$$

Multiple d	Constant $K(d)$
1.5	42
2	72
3	114
4 $(= 2^2)$	144
5	167
6 $(= 2.3)$	186
7	202
8 $(= 2^3)$	216
9 $(= 3^2)$	228
10 $(= 2.5)$	239 $\approx$ 240
11	249
12 $(= 2^2.3)$	258

Table 2

Rules for some other Multiples

Table 2 gives a whole range of rules from which many more can be devised by multiplication and division. The $K(d)$ values are integer approximations consistent with

$$n(ab) = n(a) + n(b).$$

In some cases the error committed in using a near integer having more factors may be quite small and lead to easy mental calculations. For example $K(10) \approx 240$ suggesting that at 8% compound interest 30 years would lead to a multiplication by about 10 (Note: $(1 + \frac{8}{100})^{30} \approx 10.06$).

Suppose a retiree wants to estimate the effect (i.e., the value of d) of 5% inflation for 20 years using the given table.

Now,

$$n(d) \times R = K(d)$$

$$\text{so that} \qquad K(d) = 100 \quad \text{with } R = 5 \text{ and } n(d) = 20.$$

However, $K(d) = 100$ is not in our table, so look for two values of K which differ by 100:

We have,

$$K(8) - K(3) = 102 \approx 100$$

$$\text{i.e.} \qquad K\left(\frac{8}{3}\right) = 102 \approx 100$$

$$\therefore \qquad d \approx \frac{8}{3} = 2.67.$$

and so the retiree's cost of living will have risen by a factor of 2.67 (i.e. all prices will have risen by 2.67).

This is perhaps a somewhat forced (but nevertheless valid) use of Table 2.

The actual value of d is simply

$$(1 + \frac{5}{100})^{20} \approx 2.65.$$

References

1. H. Aram "Money Matters" Unwin Paper backs, Sydney 1985 (Chapter 4)

2. D.G. Luenberger, "Investment Science" OUP, New York/Oxford (1998) (Exercise 2, Chapter 2)