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SOLUTIONS TO PROBLEMS 1057–1063

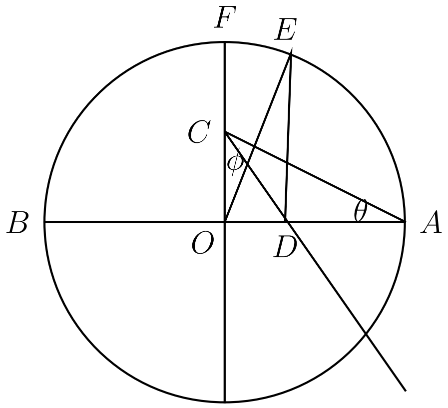
Q1057 Remember that a regular polygon has all sides equal and all angles equal.

(a) Show that the two constructions of the regular pentagon at the end of Peter Merrotsy's article (Parabola Vol.35 No.3) actually work.

(b) Starting with a regular pentagon, straight edge, and compass, construct a regular fifteen-sided polygon.

ANS. (a) In each case, $OF \perp AB$ and OA, OB, OF are radii of a circle of radius 1. The aim is to show that $\angle AOE = 2\pi/5$, i.e. that $\cos(\angle AOE) = (\sqrt{5}-1)/4$.

(i) In the diagram below, C is the midpoint of OF, CD bisects the angle $\angle OCA$ and $DE \perp AB$.



If $\angle OAC = \theta$ and $\angle OCD = \angle DCA = \phi$, then

$$2\phi + \theta = \pi/2 \qquad \text{since} \quad \Delta OAC \text{ is a right-angled triangle}$$

$$\phi = \frac{\pi/2 - \theta}{2} = \frac{\pi}{4} - \frac{\theta}{2}$$

$$\cos(\angle AOE) = OD/OE = OD = (\tan\phi)/2$$

$$2\cos(\angle AOE) = \tan\phi = \tan(\frac{\pi}{4} - \frac{\theta}{2}) = \frac{1-t}{1+t}$$

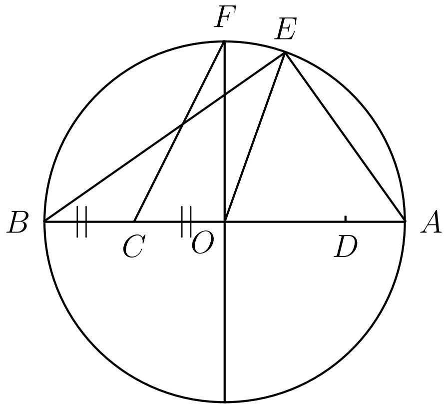
where $t = \tan(\theta/2)$, and so

$$\frac{2t}{1-t^2} = \tan\theta = \frac{OC}{OA} = 1/2.$$

Solving this quadratic, $t = \sqrt{5} - 2$ and so

$$2\cos(\angle AOE) = \frac{1 - (\sqrt{5} - 2)}{1 + (\sqrt{5} - 2)} = \frac{3 - \sqrt{5}}{\sqrt{5} - 1} = \frac{\sqrt{5} - 1}{2}.$$

(ii) In the diagram below, C is the midpoint of OB, $CD = CF$ and $BE = BD$.



Thus $BC = CO = \dfrac{1}{2}$ and so

$$CD = CF = \sqrt{CO^2 + OF^2} = \sqrt{1/4 + 1} = \sqrt{5}/2$$
$$BE = BD = BC + CD = (\sqrt{5} + 1)/2$$

Since AB is a diameter, $\angle AEB$ is a right angle and so

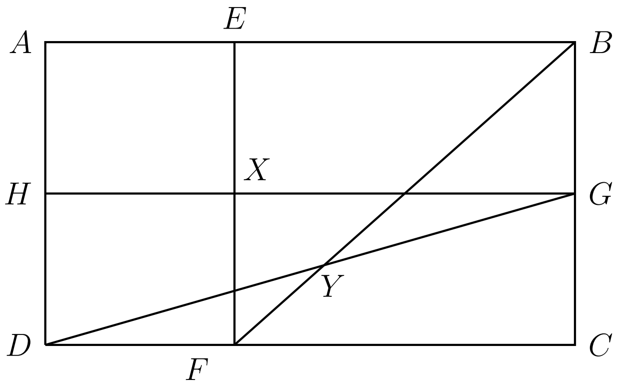
$$AE^2 = AB^2 - BE^2 = 4 - (\sqrt{5} + 1)^2/4 = (5 - \sqrt{5})/2.$$

By the cosine rule in ΔAOE,

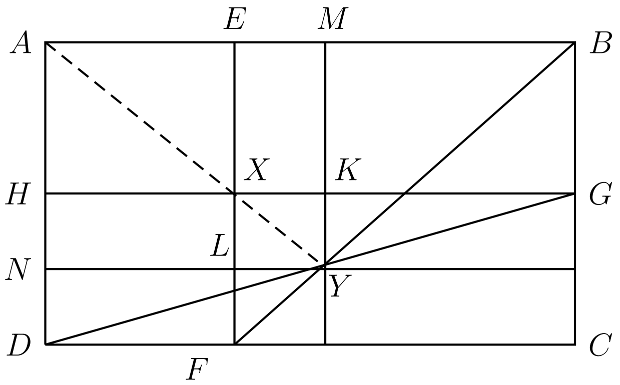
$$\begin{aligned} 2\cos(\angle AOE) &= AO^2 + OE^2 - AE^2 \\ &= 1 + 1 - (5 - \sqrt{5})/2 = (\sqrt{5} - 1)/2. \end{aligned}$$

(b) A regular pentagon can be inscribed in a circle, e.g. by finding the circumcircle of any three vertices of the pentagon. If one rotates the pentagon around the center of this circle clockwise by 60 degrees, then anti-clockwise by 60 degrees (both of which are easy to do by compass, simply by marking off a radius length along the circle from each vertex in both directions), we obtain the vertices of a regular 15-gon.

Q1058 Let $ABCD$ be a rectangle. Let EF be a line segment parallel to AD and BC which divides the rectangle $ABCD$ into two smaller rectangles $AEFD$ and $EBCF$, and let GH be a line segment parallel to AB and DC which similarly divides $ABCD$ into $ABGH$ and $HGCD$. Let X denote the intersection of EF and GH, and Y denote the intersection of BF and GD. Show that A, X, and Y are collinear.



ANS. Drop a perpendicular YL from Y to EF, and a perpendicular YK from Y to GH. Extend YK to meet AB at M, and YL to meet AD at N.



Since ΔBMY and ΔYLF are similar,

$$BM/MY = YL/LF. \tag{1}$$

Since ΔGKY and ΔYND are similar,

$$GK/KY = YN/ND. \tag{2}$$

However, we have $BM = GK$ and $LF = ND$ (opposite sides of rectangles). Thus, dividing equation (1) by equation (2), we get

$$KY/MY = YL/YN.$$

Since $YN = MA$ and $YL = KX$, we thus have

$$KY/MY = KX/MA.$$

Thus ΔKXY and ΔMAY are similar, so A, X, Y must be collinear.

Q1059 Prove that there are infinitely many positive integer solutions x, y, z to the equation

$$x^7 + y^8 = z^9.$$

ANS. Let n be any number such that n is divisible by both 7 and 8, and that $n + 1$ is divisible by 9. (There are infinitely many such numbers; anything of the form $n = 7 \times 8 \times 9 \times k + 224$ will do). Then $x = 2^{n/7}$, $y = 2^{n/8}$, and $z = 2^{(n+1)/9}$ is a solution.

Q1060 The colour of each side of a wooden cube is chosen randomly, and independently of all other sides, from one of the three colours red, green, and blue.

What is the probability that the cube has at least one pair of opposite faces which have the same colour?

ANS. The bottom face has a 2/3 chance of being a different colour from the top face. Similarly, the left face has a 2/3 chance of being a different colour from the right face, and the front face has a 2/3 chance of being different from the back face. Since all faces were coloured independently, the chance that all opposing faces have different colour is $(2/3)^3 = 8/27$. Thus the chance that at least one opposing pair of faces have the same color is $1 - 8/27 = 19/27$.

Q1061 In a tennis tournament every player plays every other player exactly once. We say that player X has *directly beaten* player Y if X plays against Y and wins; we say that player X has *indirectly beaten* player Y if X has directly beaten a player Z who has directly beaten Y. If a player has beaten all other players in the tournament (either directly or indirectly), that player is awarded a prize.

A tennis player (let's call him Pete) enters the tournament and ends up being the only player in the tournament to receive a prize. Show that Pete must have directly beaten everyone else in the tournament. (In tennis it is not possible to draw).

ANS. Pick a player other than Pete in the tournament; let's call that player X_0. We'll show that Pete must have beaten X_0 directly.

Since X_0 did not receive a prize, there must have been a player that X_0 could not beat either directly or indirectly; let's call that player X_1. Since X_0 could not beat X_1 directly, X_1 must have beaten X_0 directly. Also, if X_0 directly beat any player Y other than X_1, then X_1 must also have directly beaten Y, otherwise Y would have directly beaten X_1, and X_0 would have indirectly beaten X_1.

Thus X_1 has directly beaten everyone that X_0 has directly beaten, and has also directly beaten X_0.

If X_1 is Pete, then we are done, so suppose that X_1 is not Pete. By repeating the above argument, we may find a player X_2 who has directly beaten everyone that X_1 has beaten (including X_0), and has also directly beaten X_1.

If X_2 is Pete, then we are done, otherwise we find an X_3 who has directly beaten everyone that X_2 has beaten (including X_0), and has also directly beaten X_2.

Continuing in this fashion, we must eventually reach Pete. (We cannot cycle in an infinite loop, because each player X_{n+1} has directly beaten more players than the previous player X_n). Thus Pete has directly beaten X_0.

Q1062 Suppose that x is a real number such that

$$y = (x + \sqrt{x^2+1})^{1/3} + (x - \sqrt{x^2+1})^{1/3}$$

is an integer. Show that x is also an integer.

ANS. Taking cubes of both sides and using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$ we obtain

$$y^3 = x + \sqrt{x^2+1} + x - \sqrt{x^2+1} + 3(x + \sqrt{x^2+1})^{1/3}(x - \sqrt{x^2+1})^{1/3}y$$

which simplifies to

$$y^3 = 2x + 3(x^2 - (x^2+1))^{1/3}y$$

which simplifies further to

$$y^3 = 2x + 3y,$$

so

$$x = \frac{y(y^2-3)}{2}.$$

If y is even, we see that x is clearly an integer; if y is odd, then $y^2 - 3$ is even and again this is clearly an integer. Thus x is an integer in all cases.

Q1063 For every real number $0 \le x < 1$, let $f(x)$ be the sum of the first 1999 digits in the *binary* expansion of x. (Thus $f(0) = 0$, $f(1/2) = 1$, $f(3/4) = 2$, etc.) Compute $\int_0^1 f(x)^2\,dx$.

ANS. For each $n = 1, 2, \ldots, 1999$, let $b_n(x)$ denote the n^{th} digit in the binary expansion of x; thus

$$f(x) = \sum_{n=1}^{1999} b_n(x)$$

so that

$$f(x)^2 = \sum_{1 \le n,m \le 1999} b_n(x) b_m(x)$$

and

$$\int_0^1 f(x)^2\,dx = \sum_{1 \le n,m \le 1999} \int_0^1 b_n(x) b_m(x)\,dx.$$

Now let's compute $\int_0^1 b_n(x) b_m(x)\,dx$.

First consider the case $n = m$. Since $b_n(x)$ is always either 0 or 1, we have $b_n(x)b_n(x) = b_n(x)$, so

$$\int_0^1 b_n(x) b_n(x)\,dx = \int_0^1 b_n(x)\,dx.$$

We can split

$$\int_0^1 b_n(x)\,dx = \sum_{k=0}^{2^n-1} \int_{k/2^n}^{(k+1)/2^n} b_n(x)\,dx.$$

If k is even, then $b_n(x) = 0$ for x between $k/2^n$ and $(k+1)/2^n$, and so the integral is zero. If k is odd, then $b_n(x) = 1$ for x between $k/2^n$ and $(k+1)/2^n$, so the integral is $(k+1)/2^n - k/2^n = 2^{-n}$. Since there are only 2^{n-1} terms with k odd,

$$\int_0^1 b_n(x) b_n(x)\, dx = 2^{n-1} \times 2^{-n} = 1/2.$$

Now consider the case $n > m$. We can split $\int_0^1 b_n(x) b_m(x)\, dx$ as

$$\sum_{k=0}^{2^n-1} \int_{k/2^n}^{(k+1)/2^n} b_n(x) b_m(x)\, dx$$

and even further as

$$\sum_{k=0}^{2^n-1} \int_{l=k2^{m-n}}^{k2^{m-n}-1} \int_{l/2^m}^{(l+1)/2^m} b_n(x) b_m(x)\, dx.$$

If k is even then $b_n(x)$ is zero and if l is even then $b_m(x)$ is zero. If both k and l are odd then $b_n(x)b_m(x) = 1$, and the integral is 2^{-m}. Thus we can simplify the above as

$$\sum_{k=0(odd)}^{2^n-1} \int_{l=k2^{m-n}(odd)}^{(k+1)2^{m-n}-1} 2^{-m}.$$

There are 2^{n-1} odd numbers k from 0 to 2^{n-1}, and for each such k there are 2^{m-n-1} odd numbers l from $k2^{m-n}$ to $(k+1)2^{m-n} - 1$. Thus we obtain

$$\int_0^1 b_n(x) b_m(x)\, dx = \frac{1}{4}$$

when $n > m$. By symmetry we also have this when $n < m$.

Returning to f now, we can use the above estimates to write

$$\int_0^1 f(x)^2\, dx = \sum_{1 \le n,m \le 1999: n=m} \frac{1}{2} + \sum_{1 \le n,m \le 1999; n \ne m} \frac{1}{4}.$$

The number of terms in the first sum is 1999, and the number of sums in the second term is 1999×1998, so

$$\int_0^1 f(x)^2\, dx = \frac{1999}{2} + \frac{1999 \times 1998}{4} = \frac{1999(2 + 1998)}{4} = 1999 \times 500 = 999500.$$