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THE MATHEMATICIAN ON THE BANKNOTE: CARL FRIEDRICH GAUSS

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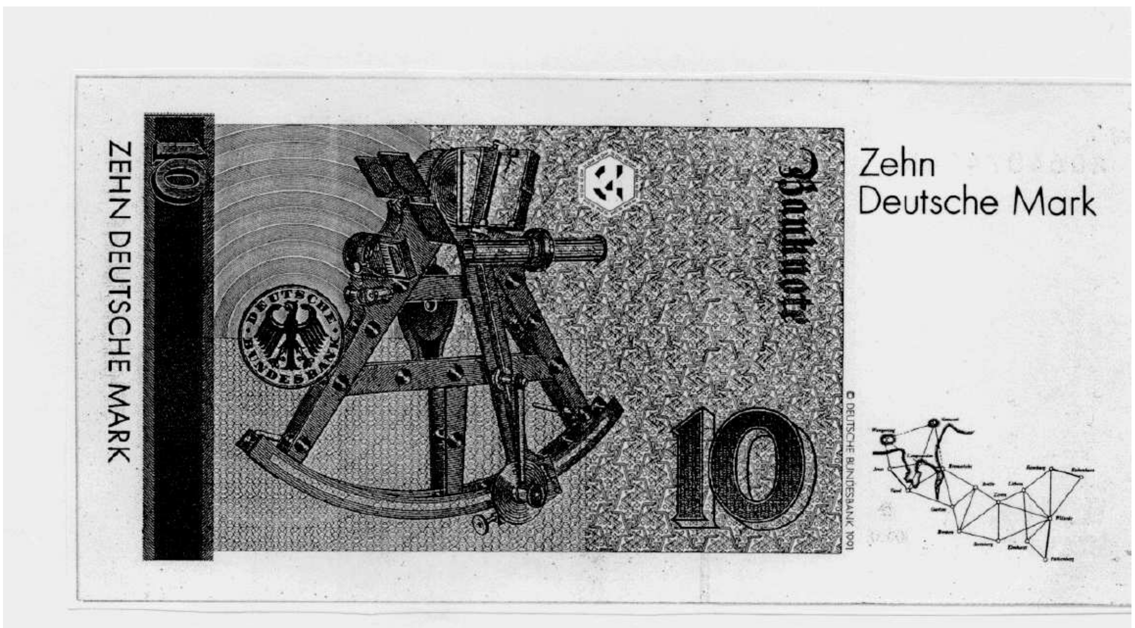
PART TWO

In the previous issue of Parabola I discussed the derivation of the normal distribution of measurement errors by the famous German mathematician Carl Friedrich Gauss in 1809. Gauss and the Normal Curve were featured on the front side a German banknote several years ago.



The reverse side of the banknote gives us a glimpse of a lesser known side of Gauss' genius – his talent as an inventor and his practical skill as a surveyor. Illustrated on the note is a heliotrope, an instrument that Gauss invented to help in his surveying, and we are shown a section of the triangular grid Gauss drew over the north of Germany as a result of the survey. What was the famous mathematician doing out in the field surveying for five years?

Gauss' interest in surveying and geodesy (the study of the shape of the Earth) went back to his youth, and he published his first paper in geodesy in 1799, when he was twenty two. He carried out his own triangulation measurements from 1803 to 1805, and worked with the French military in their survey of the German town of Brunswick,

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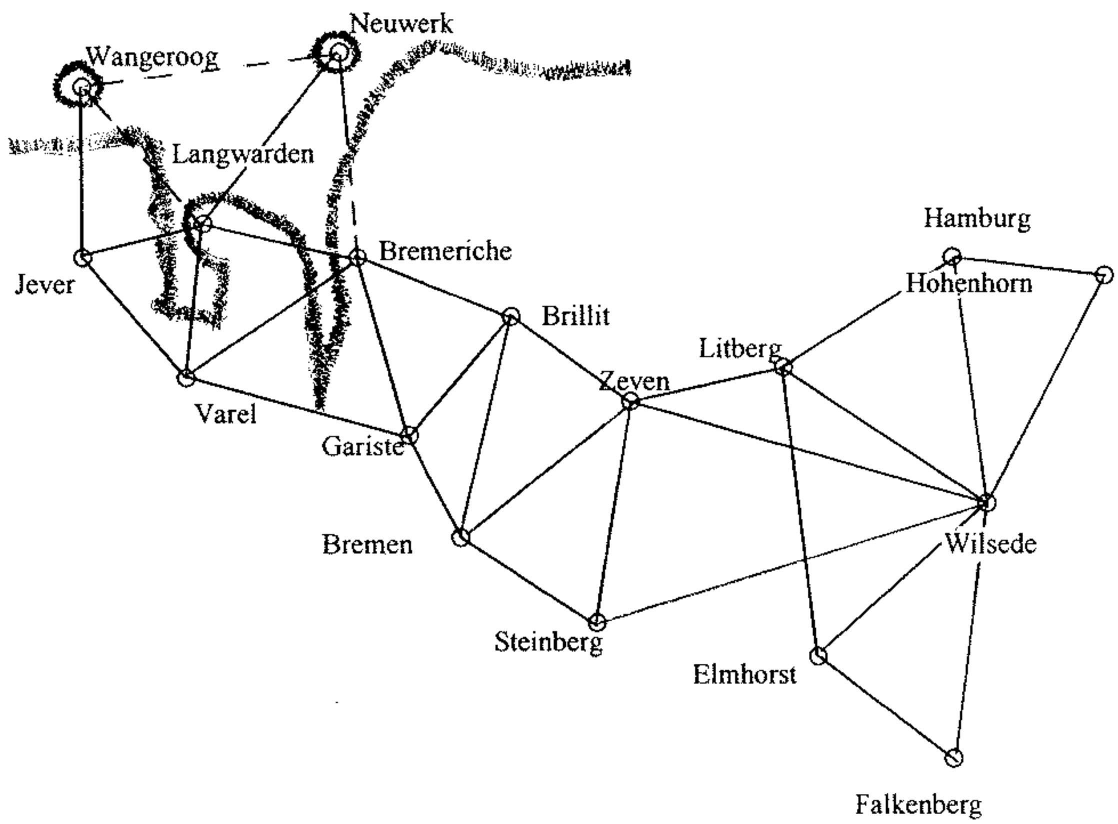
where he had been born. In 1816 his good friend Heinrich Schumacher, who was Professor of Astronomy in Copenhagen, began a survey of Denmark. It was decided to continue the Danish survey to include the neighbouring German kingdom of Hanover. It is interesting to note that at this time the region belonged to the United Kingdom of Great Britain, Ireland and Hanover, and it was King George III who commissioned Gauss to carry out the survey.

The usual way to determine the area of a piece of land is to lay a grid of triangles over the land with some sort of markers, and measure some sides (which may be kilometres in length) and angles. Then, using the Sine Rule and the Area result (Area $= \frac{1}{2}ab\sin C$) from Trigonometry the areas of the triangles can be calculated, and then summed. This technique is called triangulation. Obviously, it is important that the sides and angles are measured accurately.



Part of the triangulation of northern Germany as surveyed by Gauss, and reproduced on the reverse side of the banknote.

Gauss invented the heliotrope in order to be able to sight one point of the land from another point. The principle of the heliotrope is the same as that used by a child when he or she reflects sunlight onto distant objects with a mirror. Indeed, Gauss conceived the idea when he saw the sun reflected by a windowpane of a distant building. The heliotrope links two small vertically superimposed mirrors with a telescope. The mirrors reflect the sunlight to a chosen point several kilometres away, and with the telescope the point struck by the reflected sunlight can be easily seen, appearing like a bright shining star. As the sun moves across the sky, the mirrors can be manoeuvred by hand to rotate so that the sunlight is always reflected in the same direction. The light from the heliotrope was also used as an optical telegraph to transmit messages across vast distances.

Gauss even thought of the idea of using the heliotrope to send a light signal to the moon, in order to gain data to determine longitude accurately. He wrote, "With one hundred banked mirrors, each sixteen square feet [1.5 square metres], one would be able to send a fine heliotrope light to the moon". In effect, one could communicate with someone on the moon!

With the aid of the heliotrope Gauss was able to measure distances much longer than could be done before, and also to greater accuracy. On a bright and clear day a 15-centimetre heliotrope can be seen at 50 kilometres. With some minor improvements, the heliotrope became a very efficient instrument, and Gauss could use it without direct sunlight on days which were overcast. It was eventually superseded by improved models from 1840, and by aerial surveying in the twentieth century.

During all the fieldwork in the survey from 1821 to 1825 Gauss and his team contended with many difficulties. Transport was poor, the weather was often bad, living conditions were uncomfortable, and there was inadequate assistance and financial support. Often more than a dozen trees had to be cut down, and signal towers had to be erected in difficult places. In the evenings Gauss assessed all the data himself. His principal tool dealing with this mass of data was the method of least squares (discussed later) which he had developed. The whole experience took toll of Gauss' health, and he was involved in an accident when his carriage overturned in 1825. Though he no longer took part in the fieldwork after 1825, the surveying of Hanover continued through to 1844, with Gauss still directing the survey and making the calculations. He estimated that he had processed more than one million figures. We can be sure he would have loved to have had a modern computer.

Out of all this practical work Gauss produced some major theoretical works. In 1823 he developed his ideas on the method of least squares, and in 1828 he published his conclusions on the shape of the earth, on instrumental errors, and the calculus of observations. In 1843 and 1846 he published two papers with the title "Investigations into Subjects of Higher Geodesy". All of these works had enormous influence on the development of theoretical and practical geodesy.

In a major new direction of mathematical research he published, also in 1828, his masterful book on differential geometry, "General Investigations into Curved Surfaces". Here he discussed the problem of reproducing a curved surface on a plane or sphere, and the extended problem of representing one curved surface on any other curved surface.

Although he had some sadness in his family life, Gauss lived a full life and was honoured and respected by everyone. He died in 1855.

I wonder if one day an Australian mathematician will become so famous that he or she will be featured on an Australian banknote? Let's hope so.

The method of least squares

Gauss had investigated the method of least squares as early as 1794, but unfortunately he did not publish the method until 1809. In the meantime, the method was discovered and published in 1806 by the French mathematician Adrien-Marie Legendre, who quarrelled with Gauss about who had discovered the method first. The basic idea of the method of least squares is easy to understand.

It may seem unusual that when several people measure the same quantity, they usually do not obtain the same results. In fact, if one person measures the same quantity several times, the results will vary. What then is the best estimate for the true measurement?

The method of least squares gives a way to find the best estimate, assuming that the errors (i.e. the differences from the true value) are random. Let us consider a simple example. Suppose we measure a distance four times, and obtain the following results:

72m, 69m, 70m and 73m.

Let us denote the estimate of the true measurement by x, and form the deviations from x, namely

$$x - 72, \quad x - 69, \quad x - 70, \quad \text{and} \quad x - 73.$$

Let S be the sum of the squares of these deviations, i.e.

$$S = (x - 72)^2 + (x - 69)^2 + (x - 70)^2 + (x - 73)^2.$$

We seek the value of x that minimises the value of S.

The reader can show that $S = 4(x - 71)^2 + 10$. Hence, it can be seen that the minimum value of S is 10, when $x = 71$. (Those readers familiar with calculus may use the derivative of S to show that S has a stationary value when $x = 71$).

So 71m is the best estimate of the true measurement. Note that 71m is the mean or average of the original four measurements. It is always true that for n measurements the minimum value of S occurs when x equals the mean of the n measurements. Can you prove this?

The line of best fit

In some courses at school, students are taught to estimate the line of best fit for a set of ordered pairs. The method of least squares calculates the line of best fit, by minimising the sum of the squares of the vertical distances of the points to the line.

Consider the measurements of two quantities x and y. For example:

$$x_1 = 2, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 8, \quad x_5 = 10, \quad \text{and} \quad x_6 = 12.$$
$$y_1 = 2, \quad y_2 = 4, \quad y_3 = 4, \quad y_4 = 5, \quad y_5 = 5, \quad \text{and} \quad y_6 = 6.$$

The true values of (x_n, y_n) lie on a line $y = Ax + B$, but due to random errors in the measurements our ordered pairs do not lie on a straight line. Let us call the deviations v_n, where $v_n = Ax_n + B - y_n$, for $n = 1, 2, \ldots, 6$.

Again we form S, the sum of the squares of the deviations, and then minimise it.

$$\begin{aligned} S &= v_1^2 + v_2^2 + v_3^2 + \cdots v_6^2 \\ &= (2A + B - 2)^2 + (4A + B - 4)^2 + (6A + B - 4)^2 + \cdots + (10A + B - 5)^2. \end{aligned}$$

We now find the partial derivative of S with respect to A. This means that we differentiate S with respect to A, and treat B as if it was a constant. As with one variable, we set the derivative equal to zero.

This gives

$$182A + 21B = 103 \tag{0.1}$$

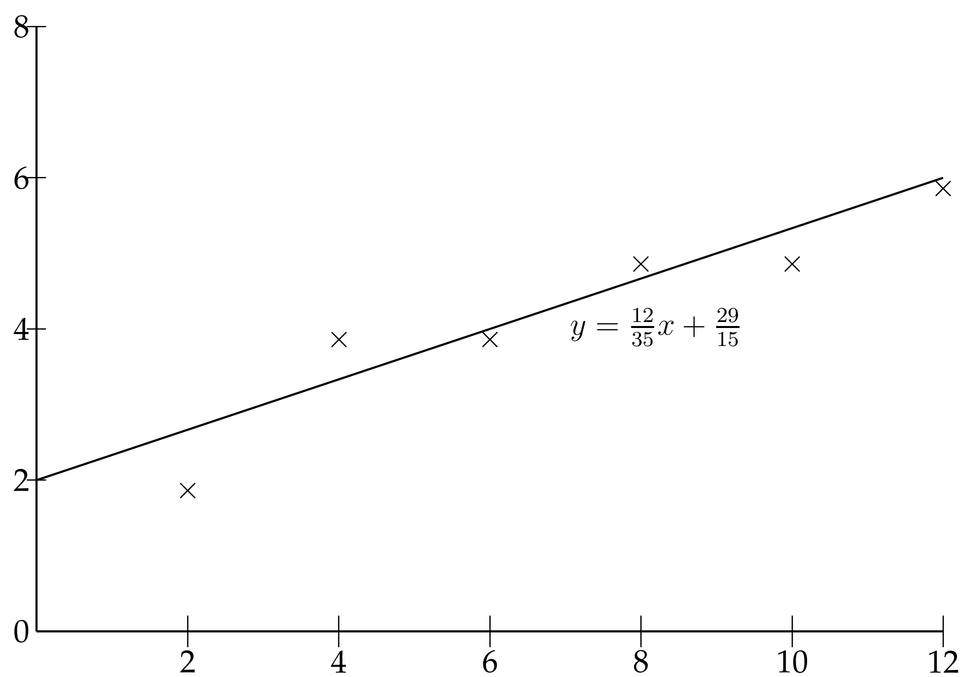
We also find the partial derivative of S with respect to B, differentiating S with respect to B, and treat A as if it was a constant, and set the derivative equal to zero.

This gives

$$21A + 3B = 13. \tag{0.2}$$

Solving (0.1) and (0.2) we have $A = \frac{12}{35}$ and $B = \frac{29}{15}$.

Hence the line of best fit is $y = (\frac{12}{35})x + (\frac{29}{15})$.



The method of least squares assumes that the errors in measurements are random and unbiased, and are distributed normally. Thus we can see a connection between the front side and the reverse side of the banknote!