Parabola Volume 36, Issue 2 (2000)

THE 72 RULE AND OTHER APPROXIMATE RULES OF COMPOUND INTEREST

Will Smith[1]

PART 2

Mortgage Loan Repayments and the Exponential Function

When people buy a home they usually have to borrow an appreciable fraction of its value from a bank or other financial institution. The bank provides the money secured by a mortgage (a legal document given by the bank for redress in the case of default by the borrower). The borrower agrees to repay the loan by constant instalments (usually monthly) over the term of the loan (often as long as 25 years). During the currency of the loan the bank changes interest on the amount unpaid at the time. The repayment instalment is adjusted so that the debt is discharged at the end of the agreed period.

First let's find how to calculate the repayments required. You might already know how to do this. There are several ways but the one below is straightforward. Suppose \$A is the original debt, \$P the payment made at the end of each of N periods (initially thought of as years) over which the loan is exactly repaid in full. Let the interest rate on unpaid monies be $R\% = 100r$ (r the fractional rate of interest) for each of the N periods.

At the start the amount owing in dollars is A. At the end of the first period after making the first payment \$P the amount owing is

$$A(1+r) - P$$

at the end of the second period it is

$$(A(1+r) - P)(1+r) - P = A(1+r)^2 - P(1 + (1+r))$$

at the end of the third period it is

$$(A(1+r)^2 - P(1 + (1+r)))(1+r) - P = A(1+r)^3 - P(1 + (1+r) + (1+r)^2)$$

At the end of the Nth period it is

$$A(1+r)^N - P(1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{N-1}).$$

[1]W.E. Smith is a mathematician retired from teaching in the Department of Applied Mathematics, UNSW.

We recognise the expression multiplying P to be a geometric series of N terms with common ratio $(1+r)$. This sums to

$$\frac{(1+r)^N - 1}{(1+r)-1} = \frac{(1+r)^N - 1}{r}$$

so the amount owing at the end of the Nth period is

$$A(1+r)^N - P\frac{(1+r)^N - 1}{r}.$$

The payment P is adjusted so that this amount is zero, i.e.,

$$A(1+r)^N - P\frac{(1+r)^N - 1}{r} = 0$$

so that

$$P = \frac{A(1+r)^N r}{(1+r)^N - 1} = \frac{Ar}{1-(1+r)^{-N}}.$$

We now modify the expressions to account for m payments per year over n years. (e.g. monthly $m = 12$). The total number of monthly payments is $N = nm$ and the fractional interest per month is $\dfrac{r}{m}$ when we choose to leave $r = \dfrac{R}{100}$ to denote the interest rate per annum. The monthly payment P_m to pay the loan is

$$P_m = \frac{\frac{Ar}{m}}{1-(1+\frac{r}{m})^{-nm}}.$$

P_m may be calculated directly on most hand-held calculators but the variation of P_m with interest rate or number of years is not transparent. We note however that P_m is the initial interest Ar/m accumulated in the first month of the loan, divided by the denominator

$$1-(1+\frac{r}{m})^{-mn} = 1 - \left[\left(1+\frac{r}{m}\right)^m\right]^{-n}$$

which accounts for the complicated interest effects over the whole course of the loan.

For m large enough $\left(1+\frac{r}{m}\right)^m$ may be approximated by the exponential function $\exp(r) = e^r$ since

$$\lim_{n\to\infty}\left(1+\frac{r}{m}\right)^m = e^r$$

which we show later in the Appendix. The approximation will work best for r small and m large. Using this approximation we obtain,

$$P_m \approx \frac{\frac{Ar}{m}}{1-e^{-nr}}.$$

How well does it work in a typical case? Take $m = 12$ (monthly), $n = 20$ (years) and $r = 0.1$ (10% p.a. interest).

$$\begin{aligned} 1-\left(1+\frac{r}{m}\right)^{-mn} &= 0.8635384 \\ 1-e^{-rn} &= 0.8646647. \end{aligned}$$

Notice the approximation does not differentiate between monthly and say weekly payment as far as the denominator is concerned. The long-term interest effects are approximately encapsulated as a simple exponential.

Appendix

Finding $\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m$.

The limit can be demonstrated in lots of ways, some easier or more elegant than others. It can be done directly from the binomial theorem but this way, although intuitive, is cumbersome to make rigorous. We have used the series for $\ln(1+x)$ in Part 1 (Parabola Vol 36 No 1) and can prove the result using it.

For a given value of r (positive, negative and zero) take m as a positive integer with $m > |r|$.

Let $E(m) = \left(1+\frac{r}{m}\right)^m$. Then $E(m) > 0$ and taking logarithms

$$\begin{aligned}
\ln E(m) &= m \ln\left(1+\frac{r}{m}\right) \\
&= m\left(\frac{r}{m} - \frac{1}{2}\left(\frac{r}{m}\right)^2 + \frac{1}{3}\left(\frac{r}{m}\right)^3 \cdots\right) \\
&= r\left(1 - \frac{1}{2}\left(\frac{r}{m}\right) + \frac{1}{3}\left(\frac{r}{m}\right)^2 \cdots\right)
\end{aligned}$$

For large m each of the terms in $\dfrac{r}{m}$ separately tends to zero but we need to be sure that their infinite sum also does.

Rearranging gives,

$$\begin{aligned}
|\ln E(m) - r| &= \frac{r}{m}\left|\frac{1}{2} - \frac{1}{3}\left(\frac{r}{m}\right) + \frac{1}{4}\left(\frac{r}{m}\right)^2 \cdots\right| \\
&\le \frac{r}{m}\left(\frac{1}{2} + \frac{1}{3}\left(\frac{|r|}{m}\right) + \frac{1}{4}\left(\frac{|r|}{m}\right)^2 + \cdots\right) \\
&< \frac{r}{m}\left(1 + \frac{|r|}{m} + \left(\frac{|r|}{m}\right)^2 + \cdots\right) \\
&= \frac{r}{m} \times \frac{1}{1-\frac{|r|}{m}}
\end{aligned}$$

summing the infinite geometric progression since $|r|/m < 1$.

As $m \to \infty$ this approaches 0. Hence $\ln E(m) \to r$ as $m \to \infty$ from which we deduce that

$$\lim_{m\to\infty} E(m) = e^r \qquad \text{for all } r.$$

A more elegant proof, but requiring integral calculus, notes that[2]

$$
\begin{aligned}
m\int_0^r \frac{dt}{t+m} &= m(\ln(m+r)-\ln m) \\
&= m\ln\left(\frac{m+r}{m}\right) \\
&= m\ln\left(1+\frac{r}{m}\right) \\
&= \ln E(m)
\end{aligned}
$$

But

$$
\begin{aligned}
m\int_0^r \frac{dt}{t+m} &= \int_0^r \left(1-\frac{t}{t+m}\right)dt \\
&= r-\int_0^r \frac{t}{t+m}dt \\
&\to r \quad \text{as} \; m\to\infty.
\end{aligned}
$$

Hence we have $\lim_{m\to\infty} \ln E(m) = r$ or $\lim_{m\to\infty} E(m) = e^r$ as above.

References

1. H. Aram, “Money Matters”, Unwin Paper backs, Sydney 1985 (Chapter 4)
2. D.G. Luenberger, “Investment Science”, OUP, New York/Oxford (1998) (Exercise 2, Chapter 2)

[2]Indebtedness to my colleagues Don Craig and Peter Brown at School of Mathematics, University of New South Wales, for this suggestion.