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UNSW SCHOOL MATHEMATICS COMPETITION 2000

SOLUTIONS

JUNIOR DIVISION

1. You have a huge pile of 1¢, 2¢, 5¢, 10¢, 20¢, 50¢ and \$1 coins. Some number of coins, N say, total X¢. Show that you can make up \$$N$ using X coins.

   Solution: Suppose the N coins that make up X¢ consist of a 1¢ coins, b 2¢, c 5¢, d 10¢, e 20¢, f 50¢ and g \$1 coins. Then

   $$a + b + c + d + e + f + g = N$$

   and

   $$a + 2b + 5c + 10d + 20e + 50f + 100g = X.$$

   Now choose a \$1 coins, $2b$ 50¢, $5c$ 20¢, $10d$ 10¢, $20e$ 5¢, $50f$ 2¢, and $100g$ 1¢ coins. The number of coins is

   $$a + 2b + 5c + 10d + 20e + 50f + 100g = X,$$

   and their value is

   $$100(a + b + c + d + e + f + g)¢ = \$N.$$

2. (a) Is it possible to number the edges of a cube $1, 2, \cdots, 12$ so that the sum of the numbers on the three edges meeting at a vertex should be the same for all eight vertices?

   (b) Is it possible to numbers the edges of a cube $1, 2, \cdots, 12$ so that the sum of the numbers on the edges of a face should be the same for all six faces?

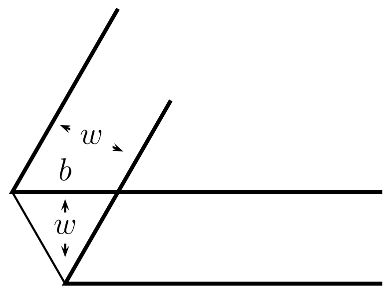
   Solution: (a) No. Suppose you can, and that the sum of the three numbers on the edges meeting at each vertex is S. Then, since there are 8 vertices and each edge is counted twice, once at each end, we have

   $$8S = 2 \times (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12) = 156,$$

   which is impossible for S an integer.

   (b) Yes. Around the top place the numbers 11, 12, 2 and 1, around the bottom 5, 3, 8 and 10, around one face 11, 4, 5, 6, around the next 12, 7, 3 and 4, around the next 2, 9, 8 and 7, and around the last face 1,6,10 and 9. The edges around every one of the six faces add to 26.

3. A long strip of paper of width w is folded as shown in the figure. What is the smallest possible area of the overlapping triangle?



Solution: The smallest possible area is $w^2/2$. The area of the triangle $A = \dfrac{1}{2}bh$, where h, the height, is w, and b, the base, is greater than or equal to w.

So $A \geq \dfrac{1}{2}w^2$.

Equality holds when the fold is at right angles, and $b = w$.

4. Let $x,\ y,\ z$ be positive numbers.

(a) Is it true that if $x + y + z \geq 3$ then $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \leq 3$?

(b) Is it true that if $x + y + z \leq 3$ then $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 3$?

Solution: (a) No. For example, if $x = \frac{1}{2},\ y = \frac{1}{2}$ and $z = 2$ then $x + y + z = 3$ while

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 4\frac{1}{2}.$$

(b) Yes. Since $x + \dfrac{1}{x} \geq 2$, and similarly for y and z,

$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 6.$$

It follows that if $x + y + z \leq 3$ then

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3.$$

5. You start with a pile of 100 stones. A "move" consists of choosing a pile containing more than two stones, throwing away one of its stones and splitting what's left of it into two smaller (not necessarily equal) piles. Can you eventually end up with a number of piles all containing three stones?

   Solution: No. There is always at least one even pile.

   Alternative Solution: No. After n moves, there are $n+1$ piles containing $100-n$ stones. If each pile has 3 stones, we have

   $$3(n+1) = 100 - n,$$

   or

   $$4n = 97,$$

   which is impossible.

6. Find the fraction with smallest denominator between $\dfrac{33}{172}$ and $\dfrac{24}{125}$.

   Solution: The answer is $\dfrac{19}{99}$.

   A fraction between $\dfrac{a}{b}$ and $\dfrac{c}{d}$ with small denominator is $\dfrac{a+c}{b+d}$.

   Here this gives $\dfrac{57}{297} = \dfrac{19}{99}$.

   We now have to show that no fraction with denominator less than 99 lies between $\dfrac{33}{172}$ and $\dfrac{24}{125}$.

   Consider the fraction $\dfrac{p}{q}$, with $q \leq 98$. The distance of $\dfrac{p}{q}$ from $\dfrac{19}{99}$ is

   $$\left|\frac{p}{q} - \frac{19}{99}\right| = \left|\frac{99p - 19q}{99q}\right| \geq \frac{1}{99q} \geq \frac{1}{98 \times 99},$$

   while

   $$\left|\frac{33}{172} - \frac{19}{99}\right| = \frac{1}{172 \times 99}, \quad \left|\frac{24}{125} - \frac{19}{99}\right| = \frac{1}{125 \times 99},$$

   so $\dfrac{p}{q}$ lies outside the interval between $\dfrac{33}{172}$ and $\dfrac{24}{125}$.

Senior

1. For any finite set A of positive integers, let $s(A)$ be the sum of the elements of A. Find the sum of $s(A)$ as A ranges over all subsets of $\{1, 2, 3, \cdots, n\}$.

   Solution: The answer is $\frac{1}{2}n(n+1) \times 2^{n-1}$. If you consider any one of the numbers $x \in \{1, 2, 3, \cdots, n\}$, x occurs in half the subsets, that is, in 2^{n-1} subsets, so it is counted 2^{n-1} times. This is true for every x. So the sum of the $s(A)$ is

   $$2^{n-1}(1 + 2 + 3 + \cdots + n) = \frac{1}{2}n(n+1)2^{n-1}.$$

2. (a) Is it possible to number the edges of a cube $1, 2, \cdots, 12$ so that the sum of the numbers on the three edges meeting at a vertex should be the same for all eight vertices?

   (b) Is it possible to numbers the edges of a cube $1, 2, \cdots, 12$ so that the sum of the numbers on the edges of a face should be the same for all six faces?

   Solution: See Junior, question 2.

3. Two circles, centres O_1 and O_2, intersect at A and B. Through A and B are drawn two parallel lines which cut the two circles forming a quadrilateral $WXYZ$.

   (a) Show that $WXYZ$ is a parallelogram, and that the length of one of its sides is equal to the length of the chord AB.

   (b) Show that the area of $WXYZ$ is maximal when WX is parallel to O_1O_2.

   Solution: (a) Let $\angle ABZ = \alpha$. Since $WX \parallel ZY$, $\angle BAW = 180^o - \alpha$ and, since $WABZ$ is a cyclic quadrilateral, $\angle AWZ = 180^o - \alpha$ as well.

   So $WABZ$ is an isosceles trapezium, with $WZ = AB$. Similarly, $XABY$ is an isosceles trapezium with $XY = AB$. So $WZ = XY = AB$. Also, $\angle ABY = 180^o - \alpha$ (ZBY is a straight angle), and $\angle AXY = \alpha$ since $XABY$ is a cyclic quadrilateral. Since $\angle AWZ$ and $\angle YXA$ are supplementary, $WZ \parallel XY$, and $WXYZ$ is a parallelogram.

   (b) Drop perpendiculars from O_1 and O_2 onto WX, meeting it at U and V respectively. Then $WA = 2UA$, $XA = 2VA$ and so $WX = 2UV$. But $UV \leq O_1O_2$. So $WX \leq 2O_1O_2$. The area of the parallelogram is less than or equal to $WX \times AB$, so is less than or equal to $2O_1O_2 \times AB$, so is maximal when $WX \parallel O_1O_2$, when the area is $2O_1O_2 \times AB$.

4. (a) Solve

   $$x + \frac{1}{y} = 5, \;\; y + \frac{1}{z} = 1, \;\; z + \frac{1}{x} = 2.$$

(b) Show that the equations

$$x + \frac{1}{y} = a, \;\; y + \frac{1}{z} = b, \;\; z + \frac{1}{x} = c$$

have a solution provided that

$$ab \neq 1, \;\; bc \neq 1, \;\; ca \neq 1, \;\; (abc - a - b - c)^2 \geq 4.$$

Solution: (a) $x = 2$, $y = \frac{1}{3}$, $z = \frac{3}{2}$. To see this,

$$\begin{aligned} z &= 2 - \frac{1}{x} = \frac{2x-1}{x}, \\ y &= 1 - \frac{1}{z} = 1 - \frac{x}{2x-1} = \frac{x-1}{2x-1}, \\ x &= 5 - \frac{1}{y} = 5 - \frac{2x-1}{x-1} = \frac{3x-4}{x-1}. \end{aligned}$$

It follows that

$$x^2 - x = 3x - 4, \;\; x^2 - 4x + 4 = 0, \;\; x = 2, \;\; y = \frac{1}{3}, \;\; z = \frac{3}{2}.$$

(b) As above,

$$\begin{aligned} z &= c - \frac{1}{x} = \frac{cx-1}{x}, \\ y &= b - \frac{1}{z} = b - \frac{x}{cx-1} = \frac{(bc-1)x - b}{cx-1}, \\ x &= a - \frac{1}{y} = a - \frac{cx-1}{(bc-1)x-b} = \frac{(abc-a-c)x - (ab-1)}{(bc-1)x-b}, \\ &(bc-1)x^2 - bx = (abc-a-c)x - (ab-1), \end{aligned}$$

and finally,

$$(bc-1)x^2 - (abc - a + b - c)x + (ab-1) = 0.$$

Similarly, the equations for y and z are

$$(ac-1)y^2 - (abc - a - b + c)y + (bc-1) = 0$$

and

$$(ab-1)z^2 - (abc + a - b - c)z + (ac-1) = 0.$$

Since $ab \neq 1$, $bc \neq 1$, $ca \neq 1$, these equations are all quadratic, and it is easily verified that the discriminant of all three is

$$\begin{aligned} (abc - a + b - c)^2 - 4(bc-1)(ab-1) &= (abc - a - b + c)^2 - 4(ac-1)(bc-1) \\ &= (abc + a - b - c)^2 - 4(ab-1)(ac-1) \\ &= (abc - a - b - c)^2 - 4. \end{aligned}$$

So all the quadratics have at least one solution (and none of the solutions is 0). For each solution for x we can find z and then y from the original equations.

5. Find the fraction with smallest denominator between $\frac{33}{172}$ and $\frac{24}{125}$.

   Solution: See Junior question 6.

6. At a trial, n witnesses (each of whom always tells the truth or always lies) sit around a large circular table. Each of them declares "I am a truthful person, but the person k seats to my right is a liar." For given k, what are the possible values of n ?

   Solution: n need only be divisible by a higher power of 2 than k. Suppose the greatest common divisor of n and k is d. Then we can write

$$n = md, \quad k = \ell d.$$

   Starting anywhere and going around the table by steps of k we will, after m steps, come back to our starting point (having gone around the table ℓ times), and when we do so we require that the number of steps in our circuit, $m,$ be **even**. This implies that n is divisible by a higher power of 2 than k.