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SOLUTIONS TO PROBLEMS 1064–1071

Q1064 The numbers $1, 2, , \ldots, 16$ are placed in the cells of a 4×4 table as shown in the left hand diagram below. One may add 1 to all numbers of any row or subtract 1 from all numbers of any column. How can one obtain the table as shown in the right hand diagram below using these operations?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

ANS. Let the number of operations applied to rows 1,2,3,4 be a_1, a_2, a_3, a_4 and the number of operations applied to columns 1,2,3,4 be b_1, b_2, b_3, b_4. Comparing the starting point and the final table we see that:

$$a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,\ a_4 = b_4,\ a_1 - b_2 = 3,\ a_1 - b_3 = 6,\ a_1 - b_4 = 9.$$

Thus, if we let a_4 be an arbitrary nonnegative integer, we have a solution to the problem, since the order in which the operations are applied does not matter. One of the solutions is $a_1 = 9,\ a_2 = 6,\ a_3 = 3,\ a_4 = 0,\ b_1 = 9,\ b_2 = 6,\ b_3 = 3$ and $b_4 = 0.$

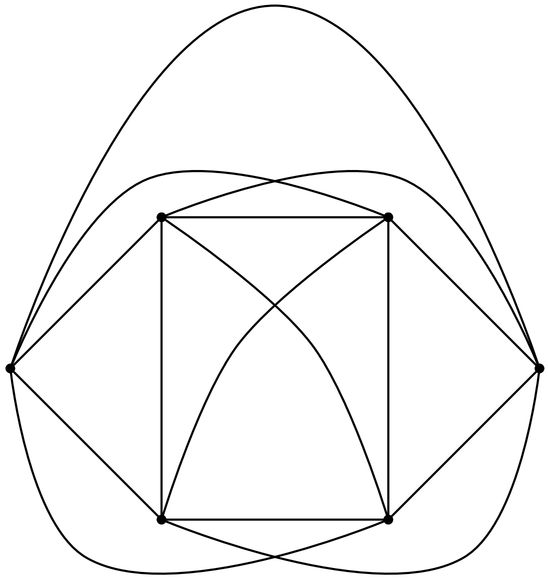
Q1065 In the land of OZ the national currency has notes of value $1, $10, $100 and $1000. Is it possible to own exactly half a million notes with a total value of $1 million?

ANS. Assume that it is possible and we have a, b, c and d notes of the four values. Then two equations must hold:

$$a + b + c + d = 500,000 \qquad a + 10b + 100c + 1000d = 1,000,000.$$

If we subtract the first equation from the second, we get $9b + 99c + 999d = 500,000$, which is impossible since 500,000 is not divisible by 9.

Q1066 The king of OZ intends to build six fortresses in the country and to connect each pair of fortresses by a two-way road. Show that it can be done so that there would be exactly three intersections and exactly two roads would cross at each intersection.
ANS. See diagram below.



Q1067 If each boy purchases a Mars bar and each girl purchases a cupcake, they would spend a total of one cent more than if each boy purchased a cupcake and each girl purchased a Mars bar. We know that there are more boys than girls. How many more boys than girls are there?

ANS. Let the number of boys be B and the number of girls be G, and the prices of a Mars bar and a cupcake be x and y cents, respectively. Then we have the equation $Bx + Gy = By + Gx + 1$, that is, $(B - G)(x - y) = 1$. However, the product of two integers can be equal to 1 only if both are equal to 1 or both are equal to -1. Since $B - G$ is positive, we conclude that it is equal to 1.

Q1068 Melbourne tram tickets have six-digit numbers (from 000000 to 999999). A ticket is called lucky if the sum of its first three digits is equal to the sum of its last three digits. How many consecutively-numbered tickets should one buy to be sure of getting at least one lucky ticket, assuming that one does not know where the sequence will start.

ANS. The answer is 1001! Note that if the first ticket we buy happens to be 000001, then the first lucky ticket we can get is 001001, that is, there is at least one case when to purchase fewer than 1001 tickets is not sufficient.

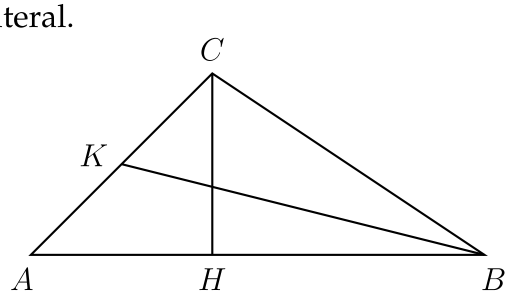
Now we have to show that 1001 is always sufficient. Write the six-digit number of the first bought ticket as AB where A represents the number formed by the first three digits and B the number formed by the last three. If $A \geq B$, we can buy $A - B \leq 1000$ tickets and obtain the lucky ticket AA. If $A < B$, the purchase of $1001 - B$ tickets leads us to the ticket $A'B'$, with $A' = A + 1$ and $B' = 0$. Then we buy an additional $A + 1$ tickets and obtain the lucky ticket $A'A'$. So we have our lucky ticket and we have bought $1002-(B-A)$ tickets. Since $B-A$ is positive we conclude that 1001 is indeed a sufficient number of tickets to buy.

Q1069 Two players play the following game on a 9×9 chessboard. They play alternatively and the first player places a black counter in a square and then the second player places a white counter in a square. The game finishes when all 81 squares are filled with counters. There are 9 rows and 9 columns and so there are either more white counters or more black counters in each row and column. Each player gets a point if they have more counters in a row or a column. So the total number of points allocated is 18. What is the highest number of points the first player can gain if both players play as well as possible?

ANS. It is easy to describe a strategy which will ensure that the first player gains ten points. She makes her first move in the central square of the board and then places her counter in the square symmetric (with respect to the center of the board) to the square the second player has filled in the previous move. This strategy guarantees that the central row and the central column belong to the first player. Further, all the other rows can be split into pairs of symmetric rows, and we see that each player gets one point from these two rows. The same is true for columns thus the first player gets exactly ten points.

Now we have to prove that the second player is able to play so that he earns at least eight points (since the total number of rows and columns on the board is 18). The strategy is for the second player to achieve a symmetric filling of the board, which, as we have seen, leaves him with the desired eight points. If the first player follows the preceding strategy, the actions of the second player are of no importance, but if the first player makes a non-symmetric move, her opponent should begin to support symmetry. If, at the beginning, the first player makes her first move in a square other than the central square, the second player can still support the necessary symmetry, and since the last move is made by the first player, she will be compelled to complete the symmetric filling of the board. Thus, we have proved that the answer is ten.

Q1070 The altitude CH and the median BK are drawn in the acute angled triangle ABC, and it is known that $BK = CH$ and $\angle KBC = \angle HBC$. Prove that the triangle ABC is equilateral.



ANS. Since the triangles BKC and CHB are obviously congruent, we have $CK =$

HB and BK is perpendicular to KC. Thus BK is an altitude, and ABC is isosceles: $AB = BC$. In addition, $\angle HBC = \angle KCB$, therefore $AB = AC$, and the proof is complete.

Q1071 The national currencies of Dillia and Dallia are called the diller and the daller, respectively. The exchange rate is such that in Dillia one diller can be exchanged for ten dallers, and in Dallia one daller can be exchanged for ten dillers. A young girl has one diller and can change her money in either country free of charge. Prove that she will never have equal numbers of dillers and dallers.

ANS. Let DIL and DAL denote the number of dillers and dallers than the girl owns, at any time, and let $S = DIL - DAL$. Initially $S = 1$. If an exchange is made in Dillia then the new value of DIL is $DIL - 1$ and the new value of DAL is $DAL + 10$, so if 1 diller is exchanged then S drops by 11. Similarly if an exchange of 1 daller is made in Dallia then S increases by 11. Hence S is always congruent to 1 modulo 11 and therefore cannot be zero.