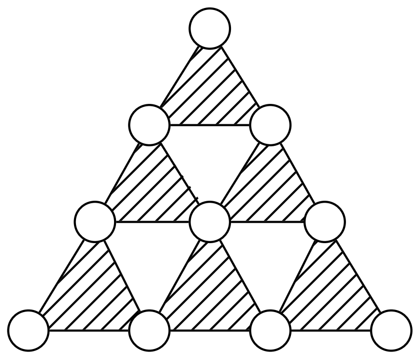
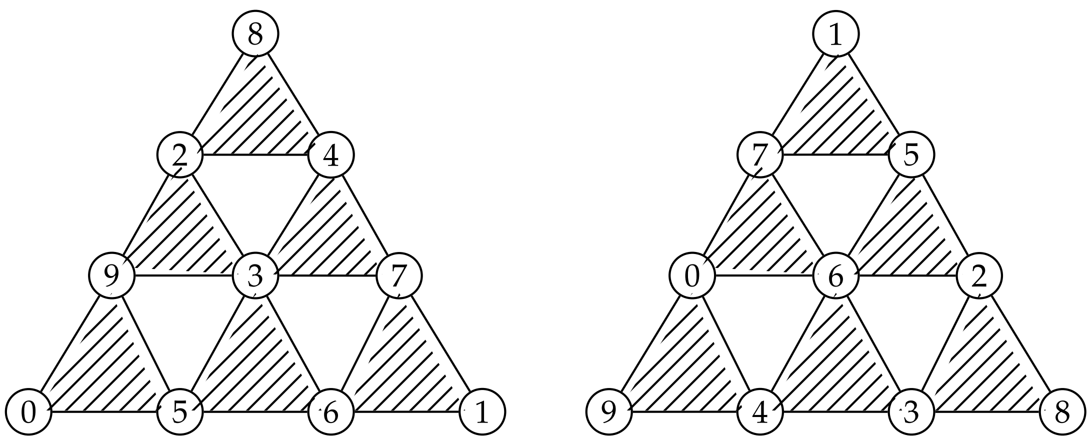
SOLUTIONS TO PROBLEMS 1072-1081

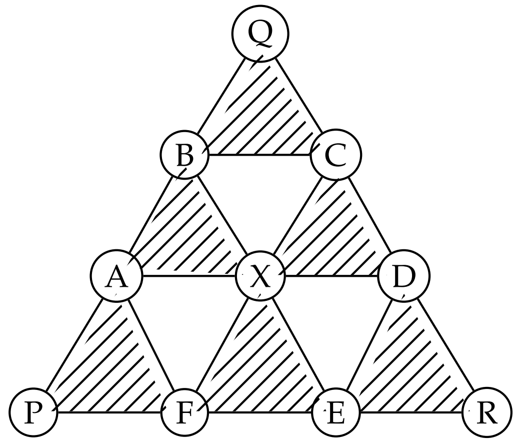
Q1072 Is it possible to fill the empty circles in the diagram below with the integers $0, 1, \ldots, 9$ so that the sum of the numbers at the vertices of each shaded triangle is the same?



ANS. The answer is yes. Moreover, there are exactly two such arrangements, not including rotations and reflections, as in the following diagram:



That is, there are 12 solutions altogether. Denote the number in the central circle by X, as in the diagram below.



Then the sum of all ten numbers is equal to $3S + X$, where S is the sum of the numbers on each shaded triangle. On the other hand, this total sum is equal to 45, so $3S+X = 45$. Thus $X = 0,\ 3,\ 6$ or 9.

If $X = 0$, then $S = 15$, and we have $A + B = C + D = E + F = 15$. But there are only two representations of 15 as a sum of two digits: $15 = 9 + 6 = 8 + 7$. Similarly, if $X = 9$ then $S = 12 = 9 + 3 + 0 = 9 + 1 + 2$ and there is no third solution, so $X = 9$ is not possible.

If $X = 3$, then $S = 14$, and we have

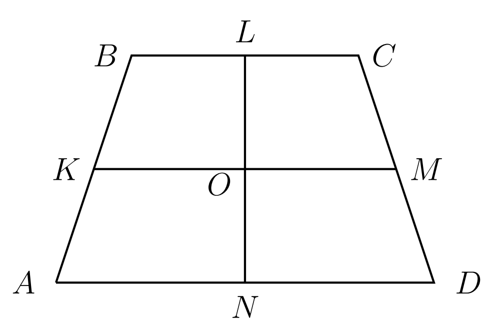
$$A + B = C + D = E + F = 11 = 9 + 2 = 8 + 3 = 7 + 4 = 6 + 5.$$

Thus the pairs (A, B), (C, D) and (E, F) can be taken to be the pairs $(9, 2)$, $(7, 4)$ and $(6, 5)$ in some order. Without loss of generality $A = 9$ and thus $B = 2$. The remaining numbers are $0,\ 1$ and 8 which must be placed in the vertices of the large triangle. Since $B = 2$, Q cannot be 0 or 1 hence $Q = 8,\ C = 4,\ D = 7$. It is now easy to see that $R = 1,\ E = 6,\ F = 5$ and $P = 0$.

The final case, $X = 6$ is analogous. Indeed, the solutions in this case can be obtained from those with $X = 3$ by simply replacing each number n by $9 - n$.

Q1073 The segments that connect the midpoints of the opposite sides of a convex quadrilateral $ABCD$ divide it into four quadrilaterals of the same perimeter. Prove that $ABCD$ is a parallelogram.

ANS. Denote the midpoints of the sides of the quadrilateral $ABCD$ by K, L, M, N and the point of intersection of the lines KM and LN by O.

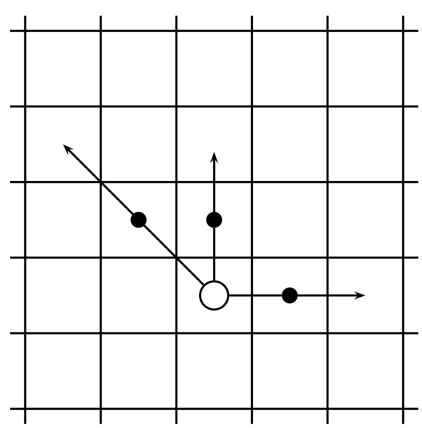


Since $BL = LC$ and $BK = KA$, KL is parallel to AC and half the length of AC. Similarly MN is parallel to and half the length of AC, so MN and KL are equal in length and parallel and $KLMN$ is parallelogram. Since the diagonals of a parallelogram bisect one another, $LO = ON$ and $KO = OM$. We now consider the perimeters of the quadrilaterals $LCMO$ and $OMDN$: $LO = ON$, $CM = MD$ and $OM = OM$ so $LC = ND$. Similarly, $BL = AN$, $BK = CM$ and $AK = DM$. So we have shown that $AB = CD$ and $BC = AD$, that is, the quadrilateral $ABCD$ is a parallelogram.

Q1074 Consider a closed path joining the 8 vertices of a cube which does not intersect itself. Prove that the path must include at least one edge of the cube.

ANS. First solution There are three types of segments which connect the vertices of a cube: edges of the cube, face diagonals and long diagonals which connect opposite vertices of the cube. The cube has six faces and since the path does not intersect itself it can include at most six face diagonals. Since the three long diagonals meet at the centre of the cube the path contains at most one long diagonal. Since $7 < 8$ the path must contain at least one edge.
Second solution Paint all the vertices of the cube in two colours, say green and gold, so that each edge joins a green point to a gold point. (Check that this can be done). The path which passes through all vertices and therefore must contain at least two segments joining a green point to a gold point. If the path contains no edges then it must include two long diagonals, which meet at the centre of the cube. This contradiction finishes the proof.

Q1075 Fifty counters are placed on a 10×10 board so that 25 of them occupy the lower-left quarter of the board and the other 25 occupy the upper-right quarter. Each counter can jump over any neighbouring square to land on the next square (if the square it lands on is free). Is it possible that after several such moves these counters will occupy only the left half of the board? (The diagram below shows three possible moves for a white counter.)



ANS. The answer is no. Paint the board green and gold like a chessboard. When a counter moves it goes from a green square to a green square or from a gold square to a gold square. So the number of green and gold squares occupied by counters does not change. At the beginning there are 26 green and 24 gold squares. If the counters were moved to the left half of the board there would be 25 counters on green squares and 25 counters on gold squares. Hence we cannot move the counters to the left half of the board.

Q1076 Evaluate

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cdots\cfrac{1}{2-\frac{1}{2}}}}}$$

where the digit 2 occurs 100 times.

ANS. Denote the fraction of the described type containing n number 2's by a_n. A little experimentation shows that $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{4}$. The natural conjecture is that $a_n = \frac{n}{(n+1)}$ which we shall prove by induction starting from $a_1 = \frac{1}{2}$. We assume that $a_{n-1} = \frac{(n-1)}{n}$ and then

$$a_n = \frac{1}{2-(n-1)/n} = \frac{n}{n+1}.$$

Hence, the answer to the problem is $\frac{100}{101}$.

Q1077 Does there exist a natural number n such that $n^n + (n+1)^n$ is divisible by 1999?

ANS. The answer is yes. If we set $n = 999$ we get

$$n^n + (n+1)^n = n^{999} + (n+1)^{999},$$

which is a multiple of $n + (n+1) = 1999$, since

$$a^n + b^n = (a+b)(a^{n-1} - \ldots + b^{n-1})$$

if n is odd.

Q1078 All cells of a 3×3 table are occupied by zeros. One may choose any 2×2 subtable and increase all the numbers in the 4 cells by 1. Prove that one cannot obtain the table shown below using these operations.

4	9	5
10	18	12
6	13	7

ANS. Each 2×2 subtable contains the central cell and exactly one of the corner cells. Thus, the number in the central cell must be equal to the sum of the numbers in the corners. But this relation does not hold for the the pictured table, and thus the result is proved.

Q1079 31 students play a game in which each student writes the numbers from 1 to 30 in any order. One of the students is designated the "leader". The leader compares his sequence with every other student's in turn and that student earns k points if his sequence and the leader's have exactly k numbers in the same position. It turns out that each of the 30 players earned a different number of points. Prove that one of the students' sequences was the same as the leader's.

ANS. Suppose one student earned 29 points. Then he would have exactly 29 numbers in the same position as the leader's. The 30th number would then also be in the same position! Hence no one can earn 29 points so each of the other 30 scores must occur, in particular someone earned 30 points and her sequence was the same as the leader's.

Q1080 Each of the natural numbers a, b, c and d is divisible by $ab - cd$, which is also a natural number. Prove that $ab - cd = 1$.

ANS. Let p denote $ab - cd$. So

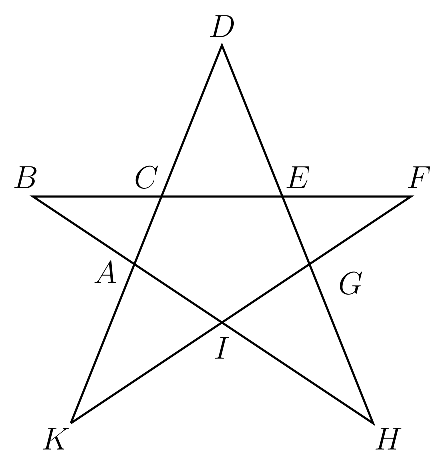
$$a = xp,\ b = yp,\ c = up, \quad \text{and} \quad d = vp,$$

where x, y, u and v are positive integers. Then

$$p = ab - cd = p^2(xy - uv),$$

and therefore p^2 divides p. Hence $p = 1$.

Q1081 Prove that it is impossible to draw a star (as shown below) so that the inequalities $AB < BC$, $CD < DE$, $EF < FG$, $GH < HI$ and $IK < KA$ all hold.



ANS. We will use the well known theorem about a triangle ABC – if $BC > AC$ then $\angle BAC > \angle ABC$. Assuming that the star could be drawn and using the fact that vertical opposite angles are opposite we obtain

$$\begin{aligned}\angle BAC &> \angle BCA = \angle DCE > \angle DEC = \angle FEG \\ &> \angle FGE = \angle HGI > \angle HIG = \angle KIA > \angle KAI = \angle BAC,\end{aligned}$$

so we have $\angle BAC > \angle BAC$, a contradiction. Hence such a star does not exist.