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Solutions 1571–1580

Q1571 Find positive integers a, b, c such that $a \leq b \leq c$ and

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{5}{26}\,.$$

SOLUTION We have

$$\frac{1}{a} < \frac{5}{26} < \frac{1}{5}\,,$$

and so $a \geq 6$. Hence, $b \geq 6$. Also,

$$\frac{bc + c + 1}{abc} = \frac{5}{26}\,,$$

so abc is a multiple of 13. Since 13 is prime, one of the numbers a, b, c must be a multiple of 13, and so $c \geq 13$. Therefore,

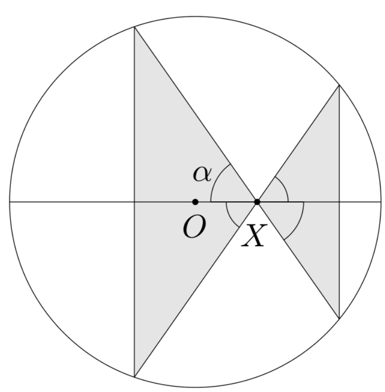
$$\frac{5}{26} \leq \frac{1}{a} + \frac{1}{6a} + \frac{1}{78a} = \frac{92}{78a}\,,$$

which implies $a \leq \frac{92}{15} < 7$, and so $a = 6$. Hence, $26(bc + c + 1) = 5abc = 30bc$, and so

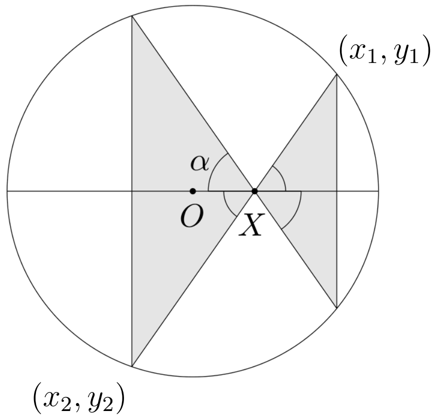
$$13 = (2b - 13)c\,.$$

We know that $c = 1$ is impossible, so $c = 13$ and $2b - 13 = 1$, giving the unique solution $a = 6$, $b = 7$, $c = 13$.

Q1572 In the following diagram, all four of the marked angles are equal.
Find the total area of the two shaded triangles in terms of the radius r of the circle, the distance $x_0 = OX$ and the angle α.



SOLUTION
Label the points where the lines meet the circle (x_1, y_1) and (x_2, y_2) as shown.



In each case we have $y_k = (x_k - x_0)\tan\alpha$. Both points are on the circle $x^2 + y^2 = r^2$ and therefore satisfy the equation

$$x^2 + (x - x_0)^2 \tan^2\alpha = r^2 \ ;$$

expanding gives

$$x^2(1 + \tan^2\alpha) - 2xx_0 \tan^2\alpha + (x_0^2 \tan^2\alpha - r^2) = 0 \ .$$

Since x_1 and x_2 are the roots of this quadratic, their sum and product can be found in terms of the coefficients:

$$x_1 + x_2 = \frac{2x_0 \tan^2\alpha}{1 + \tan^2\alpha} = 2x_0 \sin^2\alpha$$

and

$$x_1 x_2 = \frac{x_0^2 \tan^2\alpha - r^2}{1 + \tan^2\alpha} = x_0^2 \sin^2\alpha - r^2 \cos^2\alpha \ ,$$

and from this we can also calculate

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 4x_0^2 \sin^4\alpha - 2(x_0^2 \sin^2\alpha - r^2\cos^2\alpha) \ .$$

The area of a triangle is given by half the base times the altitude; treating the vertical sides as the bases gives a formula for the area, which can then be simplified:

$$\begin{aligned}
A &= y_1(x_1 - x_0) + y_2(x_2 - x_0) \\
&= [(x_1 - x_0)^2 + (x_2 - x_0)^2] \tan\alpha \\
&= [(x_1^2 + x_2^2) - 2x_0(x_1 + x_2) + 2x_0^2] \tan\alpha \\
&= [4x_0^2 \sin^4\alpha - 2x_0^2 \sin^2\alpha + 2r^2 \cos^2\alpha - 4x_0^2 \sin^2\alpha + 2x_0^2] \tan\alpha \\
&= [2x_0^2(1 - 3\sin^2\alpha + 2\sin^4\alpha) + 2r^2\cos^2\alpha] \tan\alpha \\
&= [2x_0^2(1 - 2\sin^2\alpha)(1 - \sin^2\alpha) + 2r^2\cos^2\alpha] \tan\alpha \\
&= 2\cos^2\alpha \tan\alpha [x_0^2(1 - 2\sin^2\alpha) + r^2] \\
&= (\sin 2\alpha)(x_0^2 \cos 2\alpha + r^2) \ .
\end{aligned}$$

Q1573 The diagram below shows a rectangular grid with varying row heights and column widths. Six of the sub-rectangle areas are shown: find the value of x.

x		$x+1$
$x+2$	$x+3$	
	$x+4$	$x+5$

SOLUTION Let y be the area of the middle rectangle in the top row. The ratio of areas of two rectangles having the same height is the same as the ratio of their widths. Therefore

$$\frac{x+1}{y} = \frac{\text{width of column } 3}{\text{width of column } 2} = \frac{x+5}{x+4} \,.$$

For the same reason

$$\frac{y}{x} = \frac{\text{width of column } 2}{\text{width of column } 1} = \frac{x+3}{x+2} \,;$$

multiplying these equations, the y cancels and we have

$$\frac{x+1}{x} = \frac{x+5}{x+4}\,\frac{x+3}{x+2} \,.$$

Multiplying out, we get a cubic on each side, but the x^3 terms cancel, leaving the quadratic $x^2 + x - 8 = 0$. Solving for x and rejecting the negative root gives

$$x = \frac{-1+\sqrt{33}}{2} \,.$$

Q1574 Find the smallest positive integer n with the property that, if n is divided by 61, then the 21st digit after the decimal point is 1, and the 41st digit is 9.
(**Hint**. If 10^{20} is divided by 61, then the remainder is 13.)

SOLUTION First we note that using the hint, $10^{20} = 61s + 13$ for some integer s, and therefore

$$\begin{aligned} 10^{40} &= (61s+13)^2 \\ &= 61^2s^2 + 2 \times 13 \times 61s + 13^2 \\ &= 61(61s^2 + 26s + 2) + 47 \end{aligned}$$

which leaves remainder 47 when divided by 61; and

$$\begin{aligned} 10^{60} &= (61s+13)^3 \\ &= 61^3s^3 + 3 \times 13 \times 61^2s^2 + 3 \times 13^2 \times 61s + 13^3 \\ &= 61(61^2s^3 + 3 \times 13 \times 61s^2 + 3 \times 13^2s + 36) + 1 \end{aligned}$$

which leaves remainder 1 when divided by 61. (In fact, the latter can be found without calculation from a very important result of number theory known as *Fermat's Little Theorem*.)

Now, multiplying $\frac{n}{61}$ by 10^{20} will shift the 21st digit after the decimal point to first place after the decimal point; since this digit is 1, we have

$$\frac{10^{20}n}{61} = a + x \ ,$$

where a is an integer and $0.1 < x < 0.2$. Similarly,

$$\frac{10^{40}n}{61} = b + y \ ,$$

where b is an integer and $0.9 < y < 1$. Multiplying both equations by 61 we find

$$10^{20}n = 61a + X \ , \qquad 10^{40}n = 61b + Y \ , \tag{$*$}$$

where $X = 61x$ and $Y = 61y$. Now X is an integer since it is a difference of two integers, $X = 10^{20}n - 61a$; and $6.1 < X < 12.2$. Similar reasoning for Y gives the possibilities

$$X = 7,\ 8,\ 9,\ 10,\ 11,\ 12 \quad \text{and} \quad Y = 55,\ 56,\ 57,\ 58,\ 59,\ 60.$$

Using the first of equations $(*)$ once again we have

$$10^{60}n = 10^{40} \times 61a + 10^{40}X \ ;$$

taking remainders after division by 61 and using facts mentioned above, the remainder when n is divided by 61 is the same as that for $47X$. As the first option for X is 7, the first value of $47X$ is 329 and the remainder is 24. The full set of options for n is

$$24,\ 10,\ 57,\ 43,\ 29,\ 15.$$

Similarly, the second equation from $(*)$ shows that the remainder when n is divided by 61 is one of

$$44,\ 57,\ 9,\ 22,\ 35,\ 48.$$

The only possibility common to both lists is 57. Therefore n has remainder 57 when divided by 61, and the smallest possible (positive) value for n is 57. We can check this result by calculating

$$\frac{57}{61} = 0{\cdot}93442622950819672131147540983606557377049180 3\cdots \ .$$

Q1575 Let m and n be positive integers with $m \le n$ and consider sequences

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$$

of $2n$ coin tosses, where each a_k and each b_k is either Heads or Tails. How many of these sequences have exactly m more Heads among the b_ks than among the a_ks?

SOLUTION Let H_a be the number of heads among the as and T_b the number of tails among the bs, and so on. Then we have

$$H_b = H_a + m \, , \qquad H_b + T_b = n$$

and so

$$H_a + T_b = n - m \, .$$

Therefore choosing a sequence of the desired type is the same as choosing $n-m$ places from the $2n$ places available in the sequence; letting these places be heads if they occur among the as, tails if they occur among the bs; and letting all the remaining as be tails, all the remaining bs heads. The number of choices is the binomial coefficient

$$\binom{2n}{n-m} \, , \quad \text{sometimes written} \quad {}^{2n}C_{n-m} \, .$$

Q1576 This puzzle is inspired by `www.mathsisfun.com/games/breaklock.html`, a combination of Mastermind and the Android pattern lock. We have a square pattern of nine dots, here supplemented by letters for easy reference:

$$\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix} \qquad\qquad \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

A code consists of four different dots connected by straight lines, for example $gdch$. Order is important: for example, $hcdg$ is different from $gdch$. It is not possible to skip over a dot which has not – or not yet – been used: for example, $agei$ and $agdc$ are illegal. However, it is permitted to skip over a dot that has already been used: for example, $aecg$ is allowed.

You guess the code $abcf$ and are told that two of those letters are part of the code and are in the correct position in your guess, while another one is correct but not in the correct position. You then guess $dghi$ and are told that one of these letters is correct and is in the correct position. You will be given similar information for future guesses. Can you find the code for certain in at most two more guesses?

SOLUTION Suppose that in the first guess, a and b are correct and correctly placed, while c is part of the code but not correctly placed. This means that a is the first letter in the code, b is the second, and c is not the third so it must be the fourth. The third letter must be the one in the correct place in your second guess, so it is h. Working carefully through all the possibilities gives the following potential answers:

$$\begin{matrix} 1 & 2 & 4 \\ \times & \times & \times \\ \times & 3 & \times \end{matrix} \qquad \begin{matrix} 1 & 2 & \times \\ \times & \times & 3 \\ \times & \times & 4 \end{matrix} \qquad \begin{matrix} 1 & 4 & 3 \\ \times & \times & \times \\ 2 & \times & \times \end{matrix} \qquad \begin{matrix} 1 & \times & 3 \\ \times & \times & 2 \\ \times & \times & 4 \end{matrix}$$

$$\begin{matrix} 1 & 3 & \times \\ \times & \times & 4 \\ 2 & \times & \times \end{matrix} \qquad \begin{matrix} 1 & \times & 2 \\ \times & \times & 4 \\ \times & 3 & \times \end{matrix} \qquad \begin{matrix} 4 & 2 & 3 \\ 1 & \times & \times \\ \times & \times & \times \end{matrix} \qquad \begin{matrix} \times & 2 & 3 \\ \times & \times & 1 \\ \times & \times & 4 \end{matrix}$$

$$
\begin{array}{ccc} 3 & 2 & \times \\ 1 & \times & 4 \\ \times & \times & \times \end{array}
\qquad
\begin{array}{ccc} \times & 2 & 1 \\ \times & \times & 4 \\ \times & 3 & \times \end{array}
\qquad
\begin{array}{ccc} 2 & \times & 3 \\ 1 & \times & 4 \\ \times & \times & \times \end{array}
\qquad
\begin{array}{ccc} \times & 1 & 3 \\ \times & \times & 4 \\ 2 & \times & \times \end{array}
$$

Now the first of these is impossible as the line from b to h crosses an unused dot (represented by a cross) at e; the third, fifth, sixth, tenth, eleventh and twelfth are impossible for similar reasons; which leaves five possible patterns:

$$
\begin{array}{ccc} 1 & 2 & \times \\ \times & \times & 3 \\ \times & \times & 4 \end{array}
\qquad
\begin{array}{ccc} 1 & \times & 3 \\ \times & \times & 2 \\ \times & \times & 4 \end{array}
\qquad
\begin{array}{ccc} 4 & 2 & 3 \\ 1 & \times & \times \\ \times & \times & \times \end{array}
\qquad
\begin{array}{ccc} \times & 2 & 3 \\ \times & \times & 1 \\ \times & \times & 4 \end{array}
\qquad
\begin{array}{ccc} 3 & 2 & \times \\ 1 & \times & 4 \\ \times & \times & \times \end{array}
$$

For our next guess we try $dbaf$ (other solutions may be possible). The five remaining patterns will give a response of m correctly placed, n incorrectly placed, where m, n are

$$1,\ 2; \qquad 0,\ 2; \qquad 2,\ 1; \qquad 1,\ 1; \qquad 4,\ 0$$

respectively. Since these are all different we can definitely pick the right answer on the fourth guess (unless we were lucky enough to have already picked it on the third).

Q1577 Is it possible to remove one of the numbers

$$1!, 2!, 3!, \ldots, 99!$$

in such a way that the product of the remaining 98 numbers is a square?
(**Note**: The exclamation mark denotes a factorial; for instance, $5! = 1\times2\times3\times4\times5 = 120$.)

SOLUTION No, it is not possible. Note that if we factorise a square into a product of prime numbers, then every prime number must occur an even number of times. The product of all 99 of the given numbers,

$$P = 1! \times 2! \times 3! \times \cdots \times 97! \times 98! \times 99!$$

contains the prime number 97 as a factor exactly three times, once each from $97!$ and $98!$ and $99!$. So to obtain a square, one of these must be removed. Now consider the number of times the prime 47 occurs in P. It occurs once each in $47!$ and $48!$ and… and $99!$, that is, 53 times. But additionally, it occurs once more because of the factor $94 = 2 \times 47$ in each of $94!$ and $95!$ and… and $99!$: this is another 6 times for a total of 59. So we must remove a factor which contains 47 exactly once; however, $97!$ and $98!$ and $99!$ each contain the factors 47 and 94, giving 47 twice. Therefore there is no number that can be removed from P so as to leave a square.

Q1578 Find the points of inflection, and the tangents at these points, on the graph of

$$f(x) = x^4 - 2x^3 - 36x^2 + 28x + 99\,.$$

This is a routine problem if you have studied calculus, so do it **without** calculus.

SOLUTION A polynomial $f(x)$ has a point of inflexion at $x = a$, with tangent $y = mx + c$ at that point, if

$$f(x) - (mx + c)$$

has a factor $(x - a)^3$. In this case, trying to do as little work as possible, we want

$$x^4 - 2x^3 + \cdots = (x - a)^3(x - b) = x^4 - (3a + b)x^3 + \cdots ,$$

and so $3a + b = 2$. Using this and completing the factorisation, we have

$$\begin{aligned} &x^4 - 2x^3 - 36x^2 + (28 - m)x + (99 - c) \\ &\qquad = (x - a)^3(x + 3a - 2) \\ &\qquad = x^4 - 2x^3 - (6a^2 - 6a)x^2 + (8a^3 - 6a^2)x - (3a^4 - 2a^3) . \end{aligned}$$

Equating coefficients,

$$6a^2 - 6a = 36 , \quad 28 - m = 8a^3 - 6a^2 , \quad 99 - c = -3a^4 + 2a^3 .$$

The first equation is easily solved to find possible values of a, then the others give corresponding values of m and c. Therefore the curve has

- an inflection point at $x = 3$, with tangent $y = -134x + 288$;
- an inflection point at $x = -2$, with tangent $y = 116x + 163$.

Q1579 Consider the set

$$S = \{1, 3, 12, 16, 18, 32, 36, 108, 128, 144, 162, 192, 324, \ldots\}$$

of positive integers which have no prime factors except for 2 and 3, and which when written in base 10 have first digit 1 or 3. (For example, 15 begins with a 1 but is not in S since it is 3×5; similarly, 72 is also not in S, despite that $72 = 2 \times 2 \times 2 \times 3 \times 3$, since it begins with a 7.) We can form paths from numbers in S, by defining two numbers to be adjacent in the path if their quotient is 2 or 3 or 6. An example of a path is

$$12 \longrightarrow 36 \longrightarrow 18 \longrightarrow 108 \longrightarrow 324 \longrightarrow 162 .$$

(a) Find a number in S which is not part of any path containing two or more elements.

(b) Show that any path with two or more elements can be continued indefinitely.

SOLUTION The smallest number in S which cannot be part of a path is 144. To confirm this, simply note that

$$\begin{gathered} 144 \times 2 = 288 , \quad 144 \times 3 = 432 , \quad 144 \times 6 = 864 , \\ 144 \div 2 = 72 , \quad 144 \div 3 = 48 , \quad 144 \div 6 = 24 , \end{gathered}$$

and none of the results is in S.

Now suppose that we have a path with two adjacent elements a, b from S, where $a < b$. We shall show that we can find an element $c > b$ which continues the path; by repeating the procedure, we can make the path as long as we wish. First suppose that the first digit of a is 1: then we have

$$a = \alpha 10^k ,$$

where k is an integer and $1 \le \alpha < 2$. Now b is either $2a$ or $3a$ or $6a$.

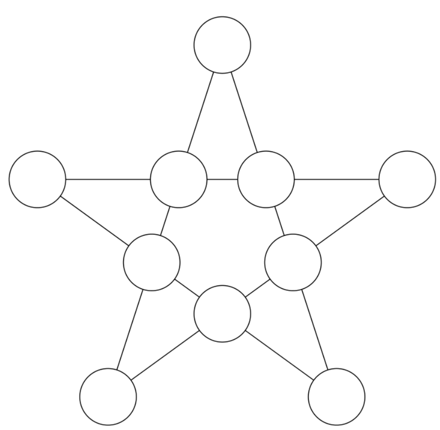
- If $b = 2a$ then $b = (2\alpha)10^k$, with $2 \le 2\alpha < 4$. Since the first digit of b cannot be 2, we have in fact $3 \le 2\alpha < 4$ and so $\frac{3}{2} \le \alpha < 2$. If $\frac{3}{2} \le \alpha < \frac{5}{3}$, choose $c = 6b > b$. Then c has prime factors 2 and 3 only; $c = 12a = (1.2\alpha)10^{k+1}$ with $1.8 \le 1.2\alpha < 2$, so c has first digit 1 and is an element of S; and the path continues from b to c since their quotient is 6. If on the other hand $\frac{5}{3} \le \alpha < 2$, choose $c = 3b$, which is in S for similar reasons (check the details for yourself).
- If $b = 3a$ then $b = (3\alpha)10^k$ with $3 \le 3\alpha < 6$ and hence $3 \le 3\alpha < 4$; so $1 \le \alpha < \frac{4}{3}$. If $1 \le \alpha < \frac{10}{9}$, choose $c = 6b$; if $\frac{10}{9} \le \alpha\frac{4}{3}$, choose $c = 3b$. Again the details are left up to you.
- If $b = 6a$ then we find $\frac{5}{3} \le \alpha < 2$; choose $c = 3b$. We have $c = 18a = (1.8\alpha)10^{k+1}$ with $3 \le 1.8\alpha < 3.6$, so c is in S.

The other possibility is that the first digit of a is 3, so that $a = \alpha 10^k$ with $3 \le \alpha < 4$.

- This time, $b = 2a$ is impossible as $2a = (2\alpha)10^k$ has first digit 6 or 7 and is not in S.
- If $b = 3a$ we can choose $c = 3b$.
- If $b = 6a$ we can choose $c = 2b$.

We have shown that a path containing two elements can always be extended by another element, and therefore, by repeating the process, can be continued for as long as we like.

Q1580 Can you place ten consecutive integers in the circles below so that every line of four integers has the same sum?



SOLUTION This solution is partly based on an elegant solution submitted to *Parabola* by Trevor Tao.

No, it is not possible. First observe that if we could do this, we could subtract the smallest of the ten numbers from all ten – which would leave all the line-sums equal – to obtain a solution in which the ten consecutive integers were $0,\ 1, \ldots,\ 9$. In this case the sum of all ten numbers would be 45; adding up the numbers in all five lines gives each one twice, total 90; and the sum in each of the five lines is on fifth of this, so 18.

Now consider the circle containing 0. It will be contained in a line with three other numbers which we denote by a; and in another line with three numbers which we denote by b. The remaining three numbers will not be on any line with 0: we denote them by c. So the three as, the three bs and the three cs are the numbers 1 to 9, once each.

Now the as add up to 18, so do the bs, therefore the cs add up to 9; the options for the cs are

$$\{2,3,4\} \quad \text{or} \quad \{1,2,6\} \quad \text{or} \quad \{1,3,5\}\ .$$

Any two of the cs form a line with one of the as and one of the bs. Likewise, each a forms a line with a b and two cs, and each b forms a line with an a and two cs.

- If the cs are $2, 3, 4$, then either as or b is 1. This 1 forms a line with two cs and a b; the sum of the two cs and the 1 is at most 8, so the b is at least 10; this is impossible
- If the cs are $1, 2, 6$, then the as are $3, 7, 8$ and the bs are $4, 5, 9$ (or *vice versa*). The line through $1, 2$ must also contain an a and a b adding up to 15: it is easy to see that this is impossible.
- If the cs are $1, 3, 5$, then the as are $2, 7, 9$ and the bs are $4, 6, 8$ (or *vice versa*). The line through $1, 3$ must also contain an a and a b adding up to 14: this is also impossible.

Since we have eliminated all possibilities, numbers cannot be placed as specified.

Bonus solution Inspired by Q1580, Dmitri Kamenetsky found and sent to *Parabola* a fascinating configuration of 10 consecutive primes with constant sum along each line. We gratefully present it here: