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The problem of the aircraft squadron revisited

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The problem of the aircraft squadron is that of determining how far some aircraft from a squadron can fly if the aircraft are able to share fuel. Alternatively, how many planes are needed for some plane to fly a given distance, and what fuel-sharing strategy would be used to achieve that?

These sort of problems attracted quite some attention for many years. Although these problems might seem obsolete today, given that long-distance records for passenger planes shared by Airbus and Boeing are over 20,000 km (a Boeing 777-200 LR broke after non-stop flying 21,601 km in about 23 hours from Hong Kong to London [1]), finding an algorithm to cover a given distance with an aircraft squadron remains an interesting recreational mathematical exercise. In [2], the problem of determining the number of aircraft in a squadron for transport aid over a certain distance is rigourously analyzed while, in [3], a graphical method is proposed for solving the problem. We present here some generalizations in an accessible manner.

Compared with the jeep problem (see e.g. [5, 6, 7]), the aircraft squadron offers more satisfaction, at least regarding the possibility to deal with large maximum distances. An interesting article on a closely-related topic, entitled "How Not to Land at Lake Tahoe!", was published in [8].

The following assumptions are made:

- All of the squadron planes are identical, having the same fuel tank capacity F and the same km/L efficiency;
- With one fuel tank, a plane can fly distance D;
- All aircraft can instantaneously share the fuel among themselves at any moment;
- Each aircraft takes off with a full tank of fuel;
- All aircraft must return home, using up all fuel upon landing.

Let TA designate the aircraft which carries the load and SA the planes which will supply the TA plane with fuel.

To this end, we raise the following questions:

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- What is the maximum distance D_{max} which can be covered by a given aircraft squadron?
- How many aircraft are necessary in order to carry out a desired operation over a certain distance?
- How long will the whole operation take?
- Which is the time moment when some SA aircraft fuels other squadron aircraft?
- When must SA aircraft take off again to meet TA at its return?

In order to solve these problems, a graphical method was in [2] chosen for the cases of one, three and five aircraft, respectively. Starting from these results, we present a simple analytical method to obtain the general solution of n aircraft and also suggest a new graphical approach.

1 Graphical triangular solution

In a time-space coordinate system (t, d), we represent the trajectory of an aircraft by a line segment that intersects the origin (at the moment $t = 0$) and has a slope that varies with the speed v. More precisely, it represents the distance travelled as a function of time (recall that the aircraft must return to the base after a certain time). This plane representation allows us to follow the flight paths efficiently over time; for instance, the returning and the departing points are the same. In fact, it is sufficient to consider a single distance d. In Figure 1, we show the simple case $n = 1$ with a single aircraft TA which has to carry its load of fuel to the destination M. With a full tank of fuel, TA will travel the maximum distance $D/2$ using $F/2$ fuel. It will then parachute the fuel for pick-up by another plane, and will finally return to the base with an empty tank.

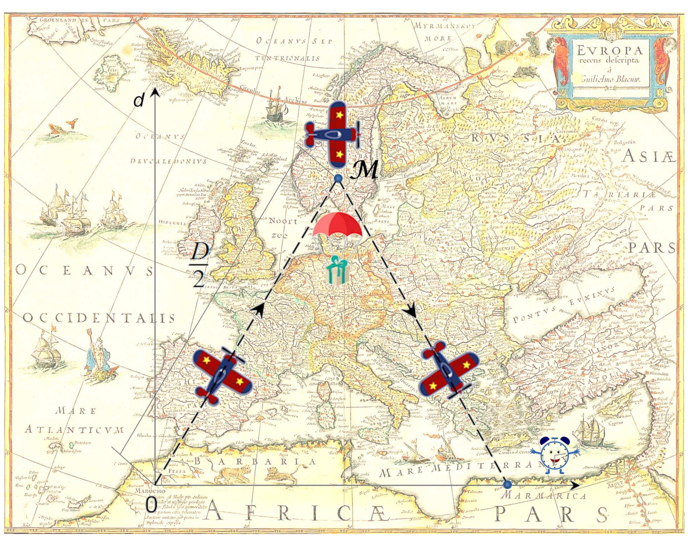

Figure 1: One aircraft ($n = 1$) in the space (t, d) (map from [4]).

The case $n = 3$ is illustrated in Figure 2a. Two aircraft take off (the TA aircraft and an SA aircraft, SA_1) at the same moment and they are in the same direction at the moment $t = 0$. After having used $1/3$ of the fuel F to travel $D/3$ of the distance, the SA_1 aircraft transfers $1/3$ of its fuel to TA at the fueling point S_1 and keeps $F/3$ fuel in order to return to the base. Next, the TA aircraft, having a full tank of fuel, will be able to travel a further distance $D/2$ in one direction and $D/2$ back. Thus, the maximum distance that it can travel is $D/3 + D/2 = 5/6D$. But, upon its return after having traveled $D/2$ from point M, it will need assistance (as the fuel tank will be empty), and so a second aircraft, SA_2, starts to fly out to meet the TA for refueling, which takes place at distance $D/3$ from the base (a point symmetric to S_1). At that moment, SA_2 transfers $F/3$ fuel to TA; thus, both aircraft have $F/3$ fuel, enough to return to the base.

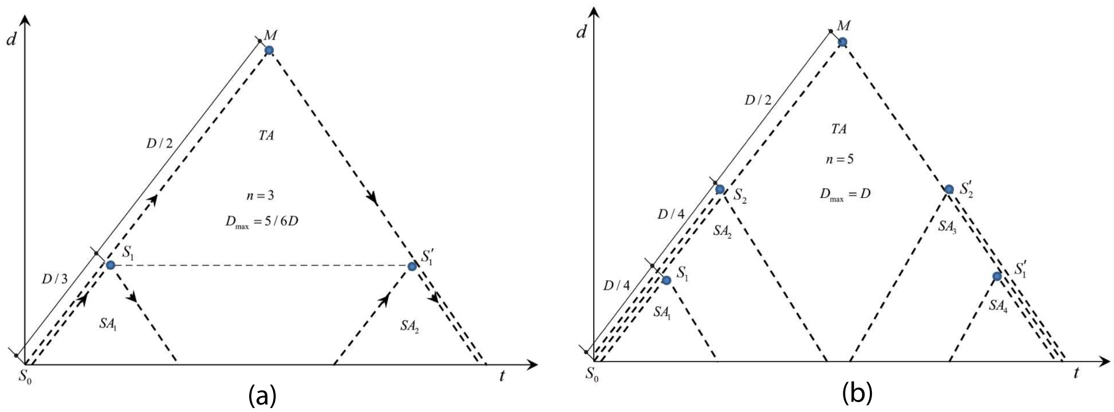

Figure 2: a) Three aircraft ($n = 3$). S_1 and S_1' are fueling points and M represents the destination point. The maximum distance flown by TA is $D_{max} = D/3 + D/2 = 5/6D$. b) Five aircraft ($n = 5$). The maximum distance is $D_{max} = D$.

The case $n = 5$ is illustrated by Figure 2b. Three aircraft take off (two SA and one TA). At distance $D/4$, at the fueling point S_1, SA_1 transfers $F/4F$ fuel to SA_2 and $F/4$ fuel to TA, after which it returns with $F/4$ fuel to the base. At $D/2$ (point S_2), SA_2 fuels TA with $F/4$ and keeps $F/2$ necessary for its return to the base. At this point, TA has a full tank of fuel, with which it travels $D/2$ to M and then $D/2$ towards the next fueling point with SA_3, at point S_2', and so on. The maximum distance covered in this case is $D_{max} = D/4 + D/4 + D/2 = D$.

2 Generalization

Cases $n = 7, 9, \ldots$ and so on are now easy to deduce. Due to the symmetry of the problem, the number of SA aircraft must be even, say $2m$, and the total number of squadron aircraft is $n = 2m + 1$. We wish to find a generalization of the method described in the previous section for $n = 2m+1$ aircraft. In this scenario, TA and m SA aircraft take off at the same time, and m SA aircraft supply the TA aircraft during its return. At each refueling point for TA, exactly one SA aircraft will contribute fuel and fly back to base. We will find the *maximum distance* covered by the squadron or, more precisely, by TA from the base to the destination point M. We will also find the *amount of fuel needed*, the *moment of fueling* between TA and each SA, the *moment of return* of each SA, the *moment of departure* for an SA to help the TA at its return, and the *total flight time*. We will prove that all of the SA aircraft can be used for both the forward journey of TA to the destination point M and the return journey of TA back to the base.

Maximum distance

Consider the first part of TA's journey and denote the m fuel-sharing points by $S_1, S_2, \ldots, S_k$ (see Figures 2a and 2b). Also, denote by AS_k the AS plane that will contribute fuel at the point S_k and then fly back. Suppose that distances between two fuel-sharing points are equal and that each plane uses f in fuel to fly from each sharing point to the next. At each point S_k, there are $m + 1 - k$ aircraft in the air, namely TA and $m - k$ SA aircraft. At the previous point S_{k-1}, SA_k had its tank filled, then consumed f to fly from S_{k-1} to S_k. It needs kf in fuel to return home and gives the rest of the fuel to TA and the other $m-k$ SA aircraft (see Figure 3 in which just two SA are considered). By design, we require that this re-fueling strategy to ensure that TA and the other $m - k$ SA aircraft will now be fully re-fueled, after having received f in fuel each. Therefore, SA_k must give them together $(m - k + 1)f$ in fuel. The fuel capacity of SA_k is $F = f + kf + (m - k + 1)f = (m + 2)f$, so

$$f = \frac{F}{m+2} = \frac{2F}{n+3}\,. \tag{1}$$

Note that this sharing of fuel means that each plane SA_k and the plane TA starts with a full tank of fuel and returns to base with no excess fuel, regardless of distance flown. In particular, each plane consumes F in fuel during the whole operation.

By symmetry, the return of TA from M involves the same fuel consumption, so the *fuel needed* for all aircraft during the total flight only depends on the number $n = 2m+1$ of aircraft and of their fuel tank capacity F:

$$F_{total} = (2m + 1)F = nF\,.$$

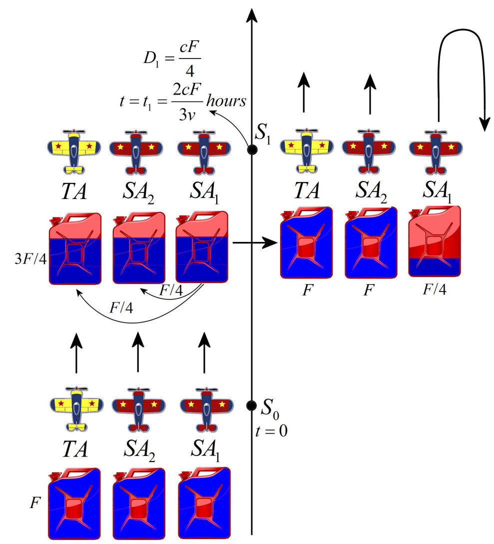

Figure 3: Starting and fueling points S_0 and S_1 respectively for two SA aircraft.

The distance D that one plane can fly on a full tank of fuel F can be expressed as $D = cF$ for some constant c measured in km/L. The maximal distance D_{max} that the squadron can fly is the distance from the base to M, or, in other words, half of the distance that TA flies in total. Since TA consumes $(mf + F + mf)/2 = mf + F/2$ in fuel to reach M, Equation (1) implies that

$$D_{max} = c(mf + F/2) = c\Big(\frac{mF}{m+2} + F/2\Big) = c\,\frac{3m+2}{2m+4}\,F$$

so

$$D_{max} = D\,\frac{3m+2}{2m+4} = D\,\frac{3n+1}{2n+6}\,. \tag{2}$$

By Formula (2), we can obtain the maximum distance for large n, which is unfortunately finite when using this method, namely $D_{max} \to \frac{3}{2}D$.

Flight time

Next, consider the moments of fueling for $SA_1, SA_2, \ldots, SA_m$, in the departing phase. The distance covered by some AS_k is $kd = 2kD/(n+2)$ where d is the distance between one fueling point S_{k-1} and the next S_k. Thus, the moments of fueling t_k can be found from the space equation: $kd = vt_k$ where v is the speed of the aircraft:

$$t_k = k\,\frac{D}{v}\,\frac{2}{n+3}\,. \tag{3}$$

The *moments of return* of SA aircraft are $2t_k = \frac{kD}{v}\frac{4}{n+3}$, and the *last SA moment of return* for SA_m is

$$2t_k = \frac{D}{v}\,\frac{2(n-1)}{n+3}\,. \tag{4}$$

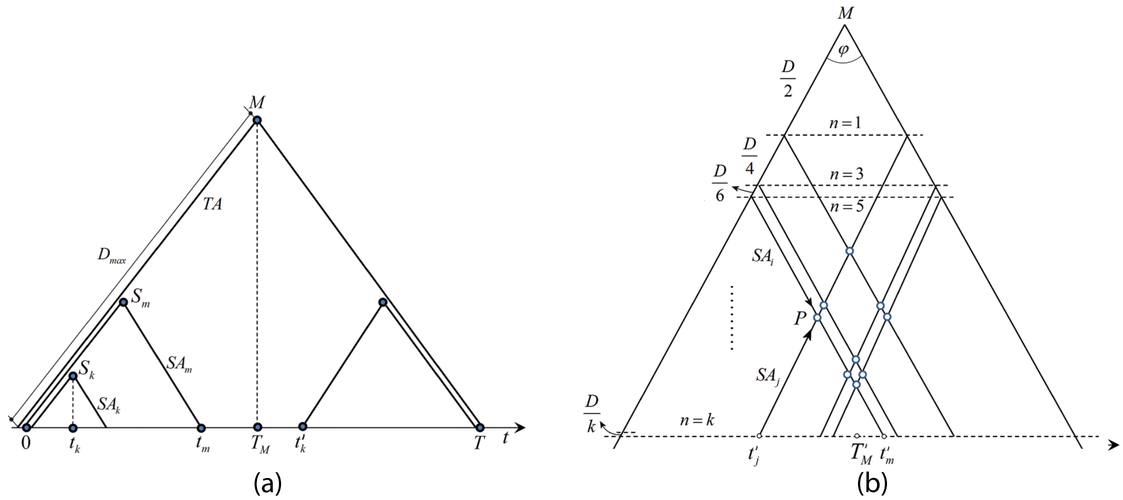

Figure 4: a) Fueling points and related times   b) Graphical rhomboid solution.

By symmetry, the time needed to fly to the destination point M and the return time to the base are the same. Therefore, the *total time* is

$$T = \frac{2D_{max}}{v} = \frac{D}{v}\frac{3n+1}{n+3}\,.$$

Next, we are interested in finding out the conditions under which some aircraft SA can help the TA both to depart and also to return. For this purpose, we need to find out if, at the destination point, there are returned SA aircraft and which could be send back to meet the TA at its return. Denote by $T_M = T/2$ the moment when the TA arrives with the fuel supply at the destination point M (see Figure 4a):

$$T_M = T/2 = \frac{D}{v}\frac{3m+2}{2(m+2)}\,. \tag{5}$$

Then

$$T_M - t_m = \frac{cF}{v} > 0\,, \tag{6}$$

which means that $T_M > t_m$. Since $t_k < t_m < T_M$, for all $k = 1, 2, \ldots, m$, at the moment T_M, all of m aircraft SA returned to the base and, therefore, can be used to help the TA at its return.

Therefore, a sufficient number of distinctive aircraft taking part in the operation is actually $m+1$, *not* $2m+1$.

Relation (6) reveals the fact that regardless of the aircraft speed v (the slope of the flight direction in the (t, d) plane), all SA aircraft returned to the base when the TA arrived at the point M.

Due to the problem symmetry, the *starting moments* t'_k for SA_k, which have to take off to help the TA, can be determined at the moments t_k:

$$t'_k = T - 2t_k = \frac{cF}{v}\frac{3n+1-4k}{n+3}\,.$$

It is easy to verify the problem symmetry, namely, the fact that all m landed aircraft AS_k, for $k = 1, 2, \ldots, m = (n-1)/2$, are ready to take off again when the TA returns from M (similarly to relation (6)) since

$$T_M - t'_k = \frac{cF}{v} \frac{-3n - 2 + 8k}{2(n+3)} < 0 \quad \text{for all} n \text{ and } k = 1, 2, 3, \ldots, (n-1)/2 \,.$$

For example, for $n = 3$, one has $T_M - t'_k = (-4 + 4k)/3 = 0$ when $k = 1$. This means that one single SA aircraft, which took off with the TA, just arrived and, after filling up "instantly" its tank, must take off to meet the TA at its return. For $n > 3$, both SA aircraft have returned before time T_M, and therefore they are ready to fly. For example, for $n = 5$, one has $T_M - t'_k = (-17 + 8k)/8 < 0$ for all $k = 1, 2$. This means that the moment t'_k, when SA_k must take off, is later than the arrival moment of the last aircraft SA_m, which helped the TA flying to the destination point M.

3 Graphical rhomboid solution

To extend D_{max} as much as possible, we propose a novel generalization of the graphical method, a *rhomboid* method. Based on the harmonic series divergence, we can increase D_{max} arbitrarily.

We draw k rhombuses in succession, starting with the upper tip of the triangle, with the side lengths $D/2, D/4, \ldots, D/2k$ (see Figure 4b). The upper triangle is closed, taking the base through the lateral kth vertex of the rhombus with a desired side length equal to D'_{max}. Now, the *maximum distance* is given by the following relation, based on the sum of the harmonic series:

$$D'_{max} = \frac{D}{2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}) \,,$$

with the number of SA aircraft being $m = 2k - 2$ in this case, in contrast to the previous method in which $m = k - 1$.

Compared with the triangular method, for a sufficiently large number of aircraft, upon the return of the SA planes on the outward journey, they can be assisted by other SA aircraft which fly to meet the TA at its return at the fueling points P (see Figure 4b) which is the crux of this method. Thus, some SA aircraft could only serve to help other (returning) SA aircraft in one or even several meeting points P.

Next, we show graphically that D_{max} (given by the relation (2)) and D'_{max} (given above) verify, counter-intuitively, the following relations (see Figure 5a):

$$D'_{max} \leq D_{max} \quad \text{for} \quad k \leq 5 \,, \tag{7}$$

and

$$D'_{max} > D_{max} \quad \text{for } k > 5 \,. \tag{8}$$

The equality holds for $k = 1$. The first inequality can be deduced numerically. For $k = 1$, it is the case of one TA aircraft (in the triangular method: $n = 1$ and $m =$

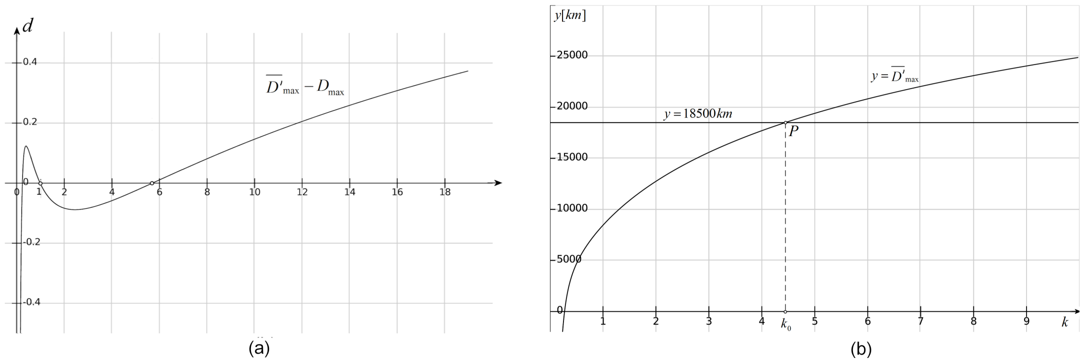

Figure 5: a) Graph of difference $\overline{D'}_{max} - D_{max}$. b) Graphical solution of equation (11).
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$k-1=0$). Thus, $D'_{max} = D/2 = D_{max}$. For $k = 2$ (in the triangular method: $n = 3$, $m = k-1 = 1$), $D'_{max} = 3/4D = 0.75D < 5/6D = 0.83D = D_{max}$. The same happens for $k = 3, 4$ and 5. For $k > 5$, the inequality changes. For example, for $k = 6$, it becomes $D'_{max} = 1.225D > D_{max} = 1.214$.

For sufficiently large k, the key points P appear. Consider a simplified hypothesis that at the return of TA, SA airplanes share fuels only at the meeting points S'_k but not at points P. Thus, for sufficiently large k, the *return time* of the last SA, t_m, becomes bigger than T_M. This, compared to the triangular method, implies the need for more SA aircraft, some of which need to take off before the landing of all SA aircraft. For example, in the case of $k = 3$ with $2 \times 3 - 2 = 4$ SA aircraft, t_m (the same as that for the triangular method, relation (4)) is $t_m = 0.8D/v$, while T_M, given by the relation (3), is $T_M = D'_{max}/v = D/(2v)(1 + 1/2 + 1/3) = 0.6D/v < t_m$.

Therefore, compared to the triangular method, more SA aircraft are necessary. Also, more fuel is necessary. However, due to the existence of points P, the situation might change.

4 Virtual application

Since 1 March 2016, the longest non-stop scheduled airline flight is Emirates flight EK448 from Dubai, United Arab Emirates to Auckland, New Zealand; like its return flight EK449, it flies a total of 14,200 km. The inaugural flight was an Airbus A380-800 but subsequent regular service flights are served by a Boeing 777-200LR: [1].

Consider the case of the Boeing 777-200LR Worldliner, with a 195,280L tank of fuel. In November 2005, it set a world distance record for a commercial aircraft non-stop flight - 21,601km - from Hong Kong to London Heathrow in about 23 hours [9].

Suppose, say, that somebody intends to carry some supply from North Pole to South Pole, the round trip distance being approximately $2 \times 18,500km = 37,000km$. Even with the above amazing capabilities, a Boeing 777-200LR could not manage to accomplish the full round trip without refueling. However, they could do so with our methods and more planes.

1. Rhomboid method. By using the Euler-Mascheroni constant, γ, which in 50 decimals[4] is (see e.g. [10]):

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

for large values of k, the partial sum of the harmonic series in (3) can be approximated, for example, as follows [11]:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} \approx \ln(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\,.$$

Thus,

$$D'_{max} \approx \overline{D'}_{max} = \frac{D}{2}\left(\ln(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\right)\,. \tag{9}$$

Therefore,

$$18,500km = \frac{21,601}{2}km\left(\ln(k) + 0.577 + \frac{1}{2k} - \frac{1}{12k^2}\right)\,. \tag{10}$$

By graphically solving this equation by finding the intersection between the horizontal line $y = 18,500$ and the graph of $\overline{D'}_{max}$ in Figure 5b), one obtains the solution $k = [k_0] = 5$. Therefore, the squadron will consist of $n = 11$ aircraft (ten SA and one TA) with $F_{total_1} = 11 \times 195,280\,L = 2,148,080\,L$. However, taking into account the P points, this number could probably be substantially reduced.

2. Triangular method. Since $18500km < 3/2 \times 21601km = lim_{n\to\infty} D_{max}$, we luckily have a solution to equation (2)[5]:

$$18,500km = \frac{3n+1}{2(n+3)}\frac{21,601}{2}km$$

which, as expected, gives a better solution $n = \lceil 2.215 \rceil = 3$, i.e., two SA aircraft (actually one single SA aircraft to be utilized twice; see Section 2) and one TA aircraft. Therefore, $F_{total_2} = 3 \times 195,280\,L = 585,840\,L \ll F_{total_1}$. However under the above-mentioned condition related to multiple potential fueling points P, F_{total_1} could drastically be reduced.

Considering that the aircraft speed is $v = 1000\,km/h$, the estimated total time of our operation (relation (5)) is

$$T = \frac{21,601km}{950km/h}\frac{3 \times 3 + 1}{3 + 1} \cong 57h$$

which, compared with the time $2 \times 25h = 50h$ of an ideal non-stop modern aircraft (calculated with a dedicated software: `http://flighttime-calculator.com/`), represents a reasonable result.

[4]Three digits are enough for our purpose.

[5]As the relation (2) shows, compared to the rhomboidal method, the triangular method cannot be used to cover any distance larger than $3/2D$.

Compared to the rhomboid method, the triangular method uses fewer SA aircraft. But, by using the rhomboid method, some of SA aircraft can serve other SA aircraft, suggesting a possible optimal solution. On the other hand, the rhomboid method allows unlimited maximum distances, theoretically of course, while the first method leads to limited maximum distances. Nevertheless, as we saw in the "virtual" application section, due to the huge capacity of the fuel tanks (for example a huge Airbus A380 can carry about 320,000L of fuel!), the limited maximum distance given by the first method can be used, even for aircraft with not-so-large fuel tanks.

However, both methods present the same drawback of flying over some countries that may not agree with these proposed algorithms.

Summary

The aircraft squadron problem is an interesting problem and is an excellent exercise for college teachers and students today. We have presented a generalization of the existing method for determining the number of aircraft to transport aid over a certain distance. Also, we have presented a new method which could provide a better solution for this challenge. The following challenging questions arise:

- What is the role of the φ angle at the point M in both methods (do not forget about the fueling points for the rhomboid method!)?
- Could one describe an equation for fuel and another for time, for the rhomboid method?
- Is the quantity of fuel for the second (rhomboidal) method greater than that for the first (triangular) method?

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