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Solutions 1581–1590

Q1581 Prove that all the numbers

$$\begin{gathered}50201905\\502019201905\\5020192019201905\\50201920192019201905\end{gathered}$$

and so on are multiples of 13.

SOLUTION We begin by confirming that the first number in the list is a multiple of 13: this is an easy calculation. Let n be any number in the list. To obtain the next number in the list we multiply by 10000, obtaining

$$502019\cdots 2019050000\ ;$$

then change the 050000 at the end to 201905 by adding 151905. That is, the next number is

$$10000n + 151905\ .$$

Once again a simple calculation shows that 151905 is a multiple of 13; thus, if n is a multiple of 13, then so is $10000n + 151905$. That is, we have shown that if any number in the list is a multiple of 13, then so is the next number. As the first number in the list is a multiple of 13, so is every number in the list.

Q1582 If the ratios

$$\frac{x+2y}{3x+5y} \quad \text{and} \quad \frac{x+3y}{4x+7y}$$

are equal, find their value.

SOLUTION If we write $r = x/y$ then the equation becomes

$$\frac{r+2}{3r+5} = \frac{r+3}{4r+7}\ . \qquad (*)$$

Multiplying out the denominators, this becomes the quadratic

$$r^2 + r - 1 = 0$$

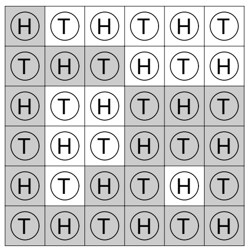
with solutions

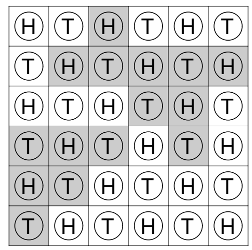
$$r = \frac{-1 \pm \sqrt{5}}{2}\ .$$

Substituting back into either side of $(*)$ provides two possible values:

$$\frac{x+2y}{3x+5y} = \frac{x+3y}{4x+7y} = \frac{3 \pm \sqrt{5}}{2}\ .$$

Q1583 Coins are placed in an $n \times n$ array, alternately heads up and tails up. A "move" means turning over some of the coins; the coins you turn must form one connected region (where "connections" may be horizontal or vertical, but not diagonal). For example, you can turn the coins indicated in grey in the left–hand diagram and it counts as one move, but to turn the coins in the right–hand diagram would require two moves.





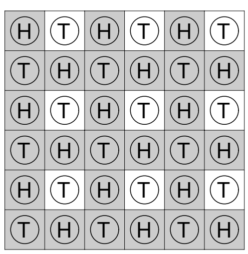
How can you get all the coins heads up in the minimum possible number of moves?

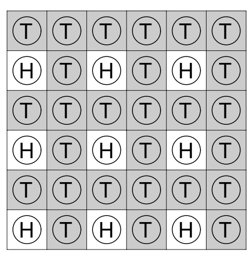
SOLUTION It's clear that the aim cannot be achieved in one move; here is a solution in two moves. We number the rows $1, 2, \ldots, n$ from the bottom up, and the columns $1, 2, \ldots, n$ from left to right. We may assume that the bottom left coin is tails.

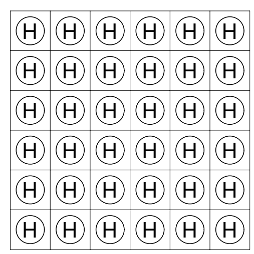
(a) Flip all coins whose row and column numbers are not both even.

(b) Flip all coins whose row and column numbers are not both odd.

The following diagrams illustrate the procedure in the case $n = 6$.





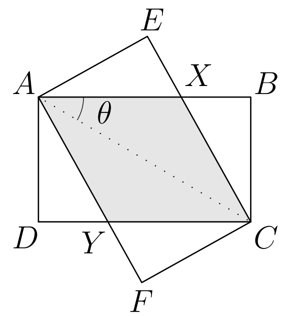


First we confirm that on any size board, each of these is a single legal move. For (i), any square in the region of flipped coins has an odd row number or column number (or both). In the first case we can travel along the row (all of whose squares are in the region) to column 1, then down the column to row 1, column 1; in the second case we travel down the column first and then along the bottom row to reach the same square; we can then reverse the procedure to reach any other square in the region. Therefore any two of the coins flipped in (i) are connected by horizontal and vertical moves, and it is a legitimate move. The argument for (ii) is virtually identical.

To check that we have obtained the desired result, observe that in the initial setup, the square in row i, column j contains a head if one of i and j is even and the other odd, a tail if i and j are both even or both odd. Therefore after moves (i) and (ii), each head has been flipped twice, so it remains a head; and each tail has been flipped once so it is now a head. Therefore all coins have ended up heads.

Q1584 The diagonal of an $a \times b$ rectangle, where $a \le b$, is placed atop the "other" diagonal of an identical rectangle. Prove that the region of overlap is a rhombus, and find its area.

SOLUTION Label the diagram as shown, where AD and AB have lengths a and b respectively.



We have $AD = AE$ by assumption; and $\angle D$ and $\angle E$ are right angles; and

$$\angle DAY = \angle DAX - \angle XAY = \angle EAY - \angle YAX = \angle EAX \ ;$$

so triangles EAX and DAY are congruent, and therefore $AX = AY$. Thus $AXCY$ is a parallelogram with two adjacent sides equal, and hence is a rhombus. Now the area of a rhombus is given by base times altitude; and we have

$$AX = \frac{AE}{\cos\angle EAX} = \frac{AE}{\cos(90^\circ - 2\theta)} = \frac{AE}{\sin 2\theta} = \frac{AE}{2\sin\theta\cos\theta} \ ;$$

and

$$\sin\theta = \frac{BC}{AC} = \frac{a}{\sqrt{a^2+b^2}} \ , \quad \cos\theta = \frac{AB}{AC} = \frac{b}{\sqrt{a^2+b^2}} \ ;$$

so the area is

$$(AX)(AD) = \frac{(AD)(AE)}{2\sin\theta\cos\theta} = a^2 \Big/ \left(\frac{2ab}{a^2+b^2}\right) = \frac{1}{2}\frac{a}{b}(a^2+b^2) \ .$$

Q1585 Three questions are asked, and three answers given:

(A):	How many of these answers are correct?	**Ans:** a
(B):	Of the first and last answers, how many are correct?	**Ans:** b
(C):	Of the last two answers, how many are correct?	**Ans:** c

Here a is a number from $\{0, 1, 2, 3\}$, and b, c are numbers from $\{0, 1, 2\}$. From these answers it is possible to tell for sure that one of the answers is correct and another is incorrect, but it is impossible to tell whether the remaining answer is correct or incorrect. What are the numbers a, b, c? Which answer was correct, and which was incorrect?

SOLUTION Consider, for example, the case where the answer to A is wrong and B and C are right. Then the correct answers to the questions are

$$a = 2\ , \quad b = 1\ , \quad c = 2\ ;$$

since A is wrong, the answers given must satisfy

$$a \neq 2\ , \quad b = 1\ , \quad c = 2$$

and so the options are

$$(a, b, c) = (0, 1, 2) \text{ or } (2, 1, 2) \text{ or } (3, 1, 2)\ .$$

Continue in this way and tabulate the options for all eight possible combinations of right and wrong answers. You will find that certain options for the given answers come up once only: $(0, 1, 2)$ is an example of this. So if the given answers were in fact $(0, 1, 2)$, we could deduce that A was wrong and the others right: that is, two correct and one incorrect: which is inconsistent with the given information. So this case must be ruled out. Then again, some options come up twice: an example is $(3, 1, 2)$, which would mean that *either* A is wrong and the others right, *or* all three answers are wrong. From these two options we could say that either A is wrong or A is wrong, so A is definitely wrong; and either B is wrong or B is right, so the status of B is unknown; and either C is wrong or C is right, so C is unknown. This gives one incorrect and two unknown, which again is inconsistent with the given informastion.

Looking through all results, we find that the only satisfactory possibility is $(a, b, c) = (2, 0, 1)$, which occurs when either B is wrong and the others right, or C is right and the others wrong. From this we find that A is unknown, B is definitely incorrect and C is definitely correct.

Q1586 A display calendar for 2019 is arranged as shown. Each column is a fixed day of the week, starting with Monday; each month is a consecutive row of dates, starting on the appropriate day of the week, with gaps at the start if required.

	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T
Jan		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31					
Feb					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28					
Mar					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
Apr	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30							
May			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31				
Jun						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30		
Jul	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31						
Aug				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31			
Sep							1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
Oct		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31					
Nov					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30			
Dec							1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31

As you can easily count, this arrangement uses 37 columns. By starting on a different day of the week, could you use fewer columns than this?

SOLUTION In a sense this problem is very easy: there are only 7 different starting days, we could just try them all and see what happens. But let's see if we can find a more insightful solution.

To avoid using 37 columns we must ensure that we never have a 31–day month preceded by 6 blank columns. The numbers of blank columns preceding these months in the given calendar are

$$1,\ 4,\ 2,\ 0,\ 3,\ 1,\ 6. \tag{$*$}$$

Shifting the start day means adding a fixed number to each of these, with any result greater than 6 being reduced by 7. We note that 5 is missing from the above numbers; if we add 1 to all of them, then 6 will be the missing number, which is what we want. (And the 6 in the present list will become 0.) So, we should start on a Sunday, giving the following 36–column calendar.

	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S
Jan			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31			
Feb						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28			
Mar						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Apr		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30					
May				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
Jun							1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Jul		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31				
Aug					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	
Sep	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30						
Oct			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31			
Nov						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
Dec	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31					

Moreover, as only one of the numbers 0 to 6 is missing in $(*)$, we cannot add a constant in such a way as to eliminate both 5 and 6. Therefore, we can never arrange the calendar so as to have 35 or fewer columns.

Q1587 Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, and suppose that

$$d = \frac{c^2}{a^2} .$$

Find b in terms of a, c, such that $f(x)$ is the square of a quadratic. Hence find all solutions of the equation

$$x^4 + 2x^3 - 20x^2 + 6x + 9 = 0 .$$

SOLUTION We'll – sort of – complete the square to get the x^4, x^3, x and constant terms right, then see what happens with the x^2 term. We have

$$(x^2 + px + q)^2 = x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2 ,$$

and so we choose

$$p = \frac{a}{2} \quad \text{and} \quad q = \frac{c}{a} .$$

This gives

$$(x^2 + px + q)^2 = x^4 + ax^3 + \Bigl(\frac{a^2}{4} + \frac{2c}{a}\Bigr)x^2 + cx + d ;$$

so we need

$$b = \frac{a^2}{4} + \frac{2c}{a} .$$

This answers the first part of the question. For the second, we note that if $a = 2$ and $c = 6$ and $d = 9$, then $d = c^2/a^2$, so the first part applies with $p = 1$ and $q = 3$ and $b = 7$. Therefore

$$\begin{aligned} &x^4 + 2x^3 - 20x^2 + 6x + 9 \\ &\qquad = (x^2 + x + 3)^2 - 27x^2 \\ &\qquad = \bigl(x^2 + (1 - 3\sqrt{3})x + 3\bigr)\bigl(x^2 + (1 + 3\sqrt{3})x + 3\bigr) \end{aligned}$$

by using the difference of two squares. We can now use the quadratic formula to find the solutions

$$x = \frac{-1 + 3\sqrt{3} \pm \sqrt{16 - 6\sqrt{3}}}{2} \quad \text{and} \quad x = \frac{-1 - 3\sqrt{3} \pm \sqrt{16 + 6\sqrt{3}}}{2} .$$

Q1588 Yesterday I tossed a fair coin 100 times. Today I tossed the coin 101 times. Find the probability that I obtained more heads today than yesterday.

SOLUTION Consider the events "I obtained more heads today than yesterday" and "I obtained more tails today than yesterday"; let their probabilities be p and q respectively. Now these events cannot both have occurred (for then I would have tossed the coin at least 102 times today); and they cannot both have failed to occur (for then I would have tossed the coin at most 100 times today). Therefore exactly one of them occurred, and so

$$p + q = 1 .$$

On the other hand, for any situation in which I tossed more heads today, imagine switching all the heads to tails, and *vice versa*, both yesterday and today: this creates an equally probable situation in which I tossed more tails today. Therefore $p = q$, and together with the previous equation, this gives $p = \frac{1}{2}$.

Q1589 A sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ starts with $a_0 = 1$; each subsequent number in the sequence is given in terms of the previous one by the rule

$$a_n = a_{n-1}^2 - 2^{2^n+1} .$$

For example, you can check that

$$a_1 = -7 \,, \; a_2 = 17 \,, \ldots, \; a_6 = -32999768863368713983 \,, \ldots .$$

Find a formula for a_n in terms of n.

SOLUTION To get rid of the large term on the right hand side of the equation, we substitute $a_n = 2^{2^n+1} b_n$. This yields

$$2^{2^n+1} b_n = \left(2^{2^{n-1}+1} b_{n-1}\right)^2 - 2^{2^n+1}$$

and so

$$2^{2^n+1} b_n = 2^{2^n+2} b_{n-1}^2 - 2^{2^n+1}$$

and so

$$b_n = 2b_{n-1}^2 - 1 .$$

This should remind you of the double–angle formula for cosine: if we can write $b_{n-1} = \cos\theta$ for some theta, then we have

$$b_n = 2\cos^2\theta - 1 = \cos(2\theta) .$$

Now $b_0 = \frac{1}{4} = \cos\theta$, where $\theta = \arccos\frac{1}{4}$; so by the double–angle formula

$$b_1 = \cos(2\theta) \,, \quad b_2 = \cos(4\theta) \,, \quad b_3 = \cos(8\theta)$$

and so on. In general, $b_n = \cos(2^n\theta)$, and therefore

$$a_n = 2^{2^n+1} \cos\left(2^n \arccos\tfrac{1}{4}\right) .$$

Q1590 Jamie (nobody you have ever heard of, just a coincidence) is a chef, and has very regular habits for lunch. He has n different recipes; the day after cooking recipe 1 he always cooks recipe 2, the day after recipe 2 is recipe 3, and so on. The only thing that is different is that on the day after recipe n, he may cook either recipe 1 or recipe 2. If you know that Jamie starts with recipe 1 on day 1, what is the first day on which, as far as you know, Jamie could be cooking any one of his n recipes?

SOLUTION Imagine two more chefs, Josie and Julie. They work though Jamie's recipes in the same way, except that Josie loves recipe 1 and will always cook it on the day after recipe n, while Julie loathes recipe 1 and will always skip from recipe n to recipe 2. Since Jamie may sometimes skip recipe 1 and sometimes not, he will never be behind Josie in the recipe schedule, nor ever ahead of Julie. The first day on which Jamie's lunch is completely indeterminate, therefore, will be the first day when Julie catches up to the recipe behind Josie. This will happen when Julie has completed (say) m circuits and has just jumped from recipe n to recipe 2, while Josie has completed $m-1$ circuits and advanced to recipe 3. Since Julie takes $n-1$ days to get from recipe 2 around to recipe 2 again, while Josie takes n days to get from recipe 3 to recipe 3, the day on which this happens will be given by

$$d = 2 + m(n-1) = 3 + (m-1)n \; .$$

Equating the two expressions for d gives

$$\begin{aligned} 2 + m(n-1) = 3 + (m-1)n \quad &\Leftrightarrow \quad mn - m + 2 = mn - n + 3 \\ &\Leftrightarrow \quad m = n - 1 \; . \end{aligned}$$

Therefore Jamie's lunch will be completely indeterminate for the first time on day $d = n^2 - 2n + 3$.