Parabola Volume 55, Issue 3 (2019)

Solutions 1591–1600

Q1591 For
$$f(x) = x^4 + 2x^3 - 7x^2 + 11 ,$$
find a line which is tangent to the graph $y = f(x)$ twice.

SOLUTION The line $y = mx + c$ is tangent to $y = f(x)$ at the point $x = a$ if and only if $(x - a)^2$ is a factor of $f(x) - (mx + c)$. In the present problem we want this to occur for two points $x = a$ and $x = b$, so we need

$$f(x) - (mx + c) = (x - a)^2(x - b)^2 ,$$

that is,
$$x^4 + 2x^3 - 7x^2 - mx + (11 - c) = [(x - a)(x - b)]^2 .$$

Since the quartic is the square of a quadratic, problem 1587 shows that we need

$$11 - c = \frac{m^2}{4} \quad \text{and} \quad -7 = 1 - m .$$

We easily find $m = 8$, $c = -5$, so the double tangent is $y = 8x-5$. We may also calculate

$$f(x) - (mx + c) = x^4 + 2x^3 - 7x^2 - 8x + 16 = (x^2 + x - 4)^2 ,$$

and so the points of tangency are

$$x = \frac{-1 \pm \sqrt{17}}{2} .$$

NOW TRY Problem 1601.

Q1592 You have an unfair coin for which heads turns up with probability $p > \frac{1}{2}$. You flip the coin repeatedly until there have been more heads than tails. How many flips, on average, does this take?

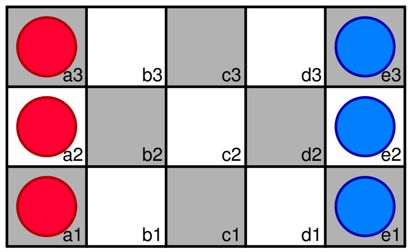
SOLUTION Suppose that it takes, on average, E flips to achieve the desired outcome. If the first flip is a head then the outcome is already achieved; and this occurs with probability p. If the first flip is a tail (probability $1 - p$) then we need to obtain more heads than tails in order to level the scores; and then do the same again to get more heads than tails overall; the number of flips is $1 + 2E$. Therefore

$$E = p(1) + (1 - p)(1 + 2E) ,$$

and solving gives
$$E = \frac{1}{2p - 1} .$$

Q1593 A 3×5 chessboard has three red counters in the leftmost column and three blue counters in the rightmost column. A counter can move to an adjacent square vertically or horizontally. Moves alternate between the two colours, and no two counters may occupy the same square simultaneously. Move the red counters into the right column and the blue counters into the left column in the minimum possible number of moves.

SOLUTION First observe that each time a red counter moves, the number of red counters on white squares increases or decreases by 1.



Since there must be one red counter on a white square both at the beginning and at the end, red must make an even number of moves; and the same applies to blue. It's clear that each counter must make 4 horizontal moves, a total of 24. Moreover in each row, at least one of the two original counters must make a vertical move in order to get out of the way of the other; this is at least 3 more moves, 27 altogether; but the total number of moves must be even, and therefore we need at least 28. For all we know at this stage, we might require even more moves than this; but trial and error gives various 28–move solutions, for example,

$$\begin{array}{lllllll}
a1-b1, & e1-d1, & b1-b2, & d1-c1, & b2-b3, & c1-b1, & c2-c1, \\
e3-d3, & a2-b2, & d3-c3, & b2-c2, & c3-b3, & c1-d1, & b3-b2, \\
a3-b3, & b2-a2, & b3-c3, & e2-d2, & c3-d3, & b1-a1, & d3-e3, \\
d2-d3, & d1-e1, & d3-c3, & c2-d2, & c3-b3, & d2-e2, & b3-a3.
\end{array}$$

Another question. For a 4×3 board a similar argument shows that we need at least 22 moves. Is a solution possible in 22 moves, or do you need more?

Q1594 Briana arranges square unit tiles in a special way. She begins with a single tile, then puts one tile on its right to form a rectangle; and then a row of tiles from left to right along the top to form a square. She then puts two tiles on the right (from bottom to top) to form a rectangle, and another row on top to form a square; and so on. For example, the diagram indicates the order of placement of the first 11 tiles.

7	8	9	
3	4	6	11
1	2	5	10

What is the perimeter of Briana's figure after the first 2019 tiles have been placed?

SOLUTION First observe that we can ignore the "missing chunk" in the top right corner, because if this chunk is filled in, the perimeter of the figure remains the same. Now suppose that n unit squares have been used, that a $k \times k$ square has been finished and the next column on the right has been started.

- If the next row on the top has *not* been started, then the number of tiles satisfies $k^2 < n \le k^2 + k$, and the perimeter $P(n)$ is that of a k by $k+1$ rectangle.
- If the next row on the top *has* been started, then the number of tiles satisfies $k^2 + k < n \le (k+1)^2$, and the perimeter $P(n)$ is that of a $k+1$ by $k+1$ rectangle.

Therefore we have

$$P(n) = \begin{cases} 4k+2 & \text{if } k^2 < n \le k(k+1) \\ 4k+4 & \text{if } k(k+1) < n \le (k+1)^2. \end{cases}$$

For the case of 2019 tiles we have

$$44 \times 45 < 2019 \le 45^2$$

and so the perimeter is

$$P(2019) = 4 \times 45 = 180\ .$$

Q1595 A regular tetrahedron has a total surface area of $16\sqrt{3}$. Find the sum of the lengths of the edges of this tetrahedron.

SOLUTION Let the side length of the tetrahedron be $2s$. Each face is an equilateral triangle with base $2s$ and height $s\sqrt{3}$ and therefore area $s^2\sqrt{3}$. There are 4 faces, so the total surface area is $4s^2\sqrt{3}$, which we know is $16\sqrt{3}$. Therefore $s = 2$, and the sum of the lengths of the six edges is $6(2s) = 24$.

Q1596 Robert has n marbles in a jar, each either blue or red. At the start there are equal numbers of blue and red marbles in the jar. Robert draws a marble from the jar twice without replacement. If the probability that both the marbles drawn are red is $\frac{2}{9}$, find n.

SOLUTION Suppose that there are m red and m blue marbles in the jar. The required probability is

$$\Bigl(\frac{m}{2m}\Bigr)\Bigl(\frac{m-1}{2m-1}\Bigr) = \frac{2}{9}\ ,$$

and solving gives $m = 5$. Therefore $n = 10$.

NOW TRY Problem 1602.

Q1597 Find the highest common factor of the integers

$$m = 2^{20} + 3^{19} \quad \text{and} \quad n = 2^{19} + 3^{20}\ .$$

SOLUTION If d is a common factor of m and n then it is also a factor of

$$3m - n = 3 \times 2^{20} - 2^{19} = 5 \times 2^{19}\ .$$

But m and n are clearly odd, so all their factors are odd; so $d = 1$ or $d = 5$. However d cannot be 5 since m is not a multiple of 5: here is one of many ways to prove this. The following table gives the last digit of $2^{k+1} + 3^k$ for $k = 1, 2, 3, \ldots$.

k	1	2	3	4	5	6	7 ...
2^{k+1}	4	8	6	2	4	8	6 ...
3^k	3	9	7	1	3	9	7 ...
$2^{k+1} + 3^k$	7	7	3	3	7	7	3 ...

Observe that this table is easy to compute: in row 2 each entry is the last digit of twice the previous entry; similarly for row 3; and row 4 is the last digit of the sum of the two rows above it. It is also clear that the first four columns will repeat indefinitely; so the last digit of $2^{k+1} + 3^k$ is never 0 or 5, and this expression cannot be a multiple of 5. Hence the only common factor, and therefore also the highest common factor, of $2^{20} + 3^{19}$ and $2^{19} + 3^{20}$ is $d = 1$.

Q1598 If the cubic polynomial

$$f(x) = x^3 + 2x^2 + 3x + 4$$

has roots a, b, c, find a cubic with roots ab, bc, ca.

SOLUTION We have

$$f(x) = (x-a)(x-b)(x-c) = x^3 + 2x^2 + 3x + 4 \ ,$$

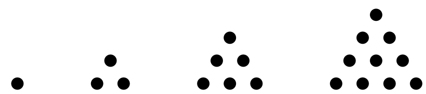
and comparing the constant terms shows that $abc = -4$. Therefore a cubic with roots ab, bc, ca is

$$\begin{aligned}
&(x-ab)(x-bc)(x-ca) \\
&\qquad = \Big(x + \frac{4}{c}\Big)\Big(x + \frac{4}{a}\Big)\Big(x + \frac{4}{b}\Big) \\
&\qquad = \frac{x}{c}\Big(c + \frac{4}{x}\Big)\,\frac{x}{a}\Big(a + \frac{4}{x}\Big)\,\frac{x}{b}\Big(b + \frac{4}{x}\Big) \\
&\qquad = \frac{x^3}{abc}\Big(a + \frac{4}{x}\Big)\Big(b + \frac{4}{x}\Big)\Big(c + \frac{4}{x}\Big) \\
&\qquad = -\frac{x^3}{abc}\Big(\Big(-\frac{4}{x}\Big) - a\Big)\Big(\Big(-\frac{4}{x}\Big) - b\Big)\Big(\Big(-\frac{4}{x}\Big) - c\Big) \\
&\qquad = \frac{x^3}{4} f\Big(-\frac{4}{x}\Big) \\
&\qquad = \frac{x^3}{4}\Big(-\frac{64}{x^3} + \frac{32}{x^2} - \frac{12}{x} + 4\Big) \\
&\qquad = x^3 - 3x^2 + 8x - 16 \ .
\end{aligned}$$

Q1599 The **triangular numbers** are

$$T_1 = 1,\ \ T_2 = 1+2 = 3,\ \ T_3 = 1+2+3 = 6,\ \ T_4 = 1+2+3+4 = 10$$

and so on; we also define $T_0 = 0$. They can be illustrated as the numbers of dots which can be formed into equilateral triangular shapes:



Let n be a positive integer. Prove that the number of ways of writing n as a difference of two triangular numbers, $n = T_a - T_b$, is equal to the number of odd factors of n.

SOLUTION We can obtain a formula for the triangular numbers by adding up an arithmetic progression:

$$T_a = 1 + 2 + \cdots + a = \frac{a(a+1)}{2}\ .$$

So if n is a difference of two triangular numbers, then we have

$$n = T_a - T_b = \frac{a(a+1)}{2} - \frac{b(b+1)}{2} \quad \text{with } a > b \geq 0$$

and so

$$8n = 4a(a+1) - 4b(b+1) = (2a+1)^2 - (2b+1)^2 = (2a-2b)(2a+2b+2)\ .$$

Therefore,

$$2n = (a-b)(a+b+1)\ .$$

Now the sum of these last two factors is $(a-b)+(a+b+1) = 2a+1$, which is odd; so one of the factors is even and the other is odd. Therefore, for every expression $n = T_a - T_b$ we can find an odd factor of $2n$. Conversely, if p is an odd factor of $2n$ then we have $2n = pq$, where q is even. If l is the larger of p and q, and s is the smaller, we can choose non–negative integers

$$a = \frac{l+s-1}{2}\ , \quad b = \frac{l-s-1}{2}\ .$$

Using the same algebra as above, we then have

$$(a-b)(a+b+1) = sl = 2n$$

so

$$n = T_a - T_b$$

and we have found an expression of n as the difference of two triangular numbers. Thus, the number of ways of writing n as a difference of two triangular numbers is equal to the number of odd factors of $2n$. Finally, an odd number is a factor of $2n$ if and only if it is a factor of n, and this completes the proof.

Q1600 A social network has $2n$ members. Any two members are either friends or not. There are no three members p, q, r that are all friends with each other.
Prove that there are at most n^2 unordered friend–pairs.
(Here, $\{p, q\}$ and $\{q, p\}$ count as the same friend-pair.)

SOLUTION Denote the set of all members of the social network by N. Call a set of network members "unfriendly" if no two of them are friends: note that this doesn't mean people in the set have no friends at all, it just means they have no friends in the set. Let U be the largest possible unfriendly set in N, and let u be the number of people in U. Observe that, in particular, given any specific member m of N, the friends of m form an unfriendly set: for if any two of them were friends, then those two together with m would form a "friend–triple". It follows that each member has at most u friends.

Now any friend–pair must include at least one person not in U. There are $2n-u$ such people, and each has at most u friends, so there are at most $u(2n-u)$ friend–pairs. But as u varies, this is a quadratic which has a maximum when $u = n$, and the maximum value is $n(2n-n) = n^2$. Thus there are at most n^2 friend–pairs.