Parabola Volume 56, Issue 1 (2020)

A geometric approach to Fibonacci and Lucas sequences

Radu-Ioan Mihai[1]

Mathematics is the art of giving the same names to different things.
– Henri Poincaré, *Science et Mēthode*, 1908.

1 Introduction

Fibonacci numbers are well-known and well-studied, as are the related Lucas numbers. The usual approach to studying these two sequences of numbers is usually algebraical and combinatorial. In this article, we would like to present a geometric interpretation of these interrelated sequences and their unique properties.

Let us first define these numbers. The *Fibonacci sequence* $(F_n)_{n\geq 0}$ is the sequence of numbers

$$F_0 = 0\,, \quad F_1 = 1\,, \quad F_2 = 1\,, \quad F_3 = 2\,, \quad F_4 = 3\,, \quad F_5 = 5\,, \quad F_6 = 8\,, \quad \ldots\,,$$

defined recursively by the formula

$$F_{n+2} = F_{n+1} + F_n\,. \tag{1}$$

For instance, $8 = 5 + 3$, so $F_6 = F_5 + F_4$. The *Lucas sequence* $(L_n)_{n\geq 0}$ is defined by the same recursive formula as Fibonacci sequence ($L_{n+2} = L_{n+1} + L_n$) but the starting term ($L_0 = 2$) is different:

$$L_0 = 2\,, \quad L_1 = 1\,, \quad L_2 = 3\,, \quad L_3 = 4\,, \quad L_4 = 7\,, \quad L_5 = 11\,, \quad L_6 = 18\ldots\,.$$

These sequences of numbers have been studied extensively [5, 6] and there are many examples of nature's rules following their recursion formula, including for instance the shapes of spiral galaxies, such as Messier 74, and hurricanes, such as Hurricane Irene (see, for example [1]).

Many properties of the two sequences are similar but not all. We will look at some of the interesting ways in which the Fibonacci numbers and the Lucas numbers are related. Our approach will primarily be geometric and visual.

[1]Radu-Ioan Mihai is a 11th grade student at the "Tudor Vianu" National College of Computer Science, Bucharest, Romania (rimihai2001@gmail.com).

2 Known relations

Before looking at the geometric ways in which to relate the Fibonacci numbers and the Lucas numbers, let us first consider some simple and well-known algebraic relations between these numbers. By solving recurrence equation (1) with respect to initial conditions $F_0 = 0$ and $F_1 = 1$, and solving similarly for Lucas numbers with initial conditions $L_0 = 2$ and $L_1 = 1$, we find that

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \varphi^{-n}\right) \qquad \text{and} \qquad L_n = \varphi^n + \varphi^{-n}\,, \tag{2}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the famous *golden ratio*. The expressions above, the first of which is called *Binet's Formula*, can be used to prove the following relationships between the Lucas and Fibonacci sequences. We will however prove these more directly.

Proposition 1.

$$L_n = F_n + 2F_{n-1}\,.$$

Proof. Certainly, $L_1 = 1 = 1 + 2 \times 0 = F_1 + 2F_0$ and $L_2 = 3 = 1 + 2 \times 1 = F_2 + 2F_1$. Assume that $L_n = F_n + 2F_{n-1}$ for all values up some integer $n \geq 1$. Then by (1),

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ &= (F_n + 2F_{n-1}) + (F_{n-1} + 2F_{n-2}) \\ &= (F_n + F_{n-1}) + 2(F_{n-1} + F_{n-2}) \\ &= F_{n+1} + 2F_n\,. \end{aligned}$$

By induction, $L_n = F_n + 2F_{n-1}$ is true for all $n \geq 1$, and (1) concludes the proof. □

Many other relations, like that in Proposition 1, exist between Fibonacci numbers and Lucas numbers. For instance, by applying (1) to Proposition 1, we find that

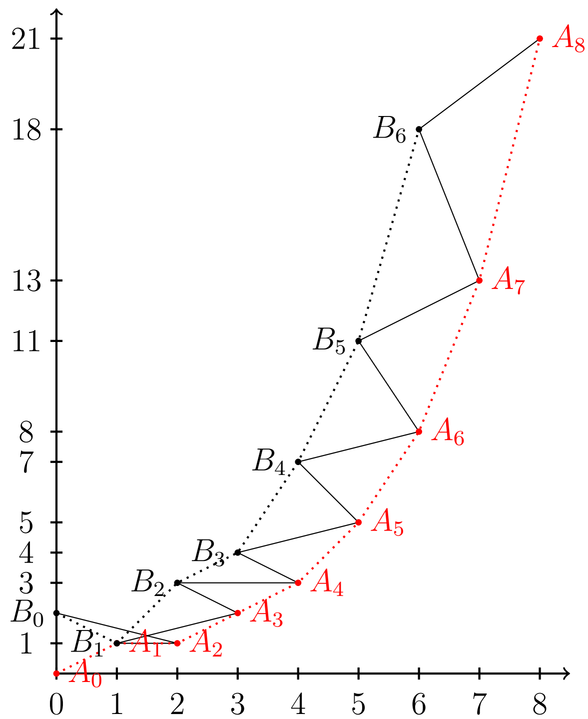
$$\begin{aligned} L_n &= 2F_n + F_{n-3} && (3) \\ &= F_n + F_{n-1} + F_{n-2} + F_{n-3} && (4) \\ &= F_{n+1} + F_{n-1}\,. && (5) \end{aligned}$$

You can find these and other algebraic relations between Fibonacci numbers and Lucas numbers in [7], including the following:

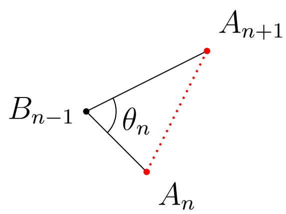
$$\begin{aligned} F_n &= \frac{1}{5}\left(L_{n+1} + L_{n-1}\right) && (6) \\ L_n &= F_{n+2} - F_{n-2} && (7) \\ F_{2n} &= L_n F_n\,. && (8) \end{aligned}$$

3 A geometrical approach

Let us now draw the points $A_n(n, F_n)$ and $B_n(n, L_n)$ as follows.



We consider the triangles with points A_n, B_{n-1} and B_{n+1} and angle $\theta_n = \angle(A_n B_{n-1} A_{n+1})$:



From (5), (7) and (1), we see that

$$\begin{aligned} \overrightarrow{B_{n-1}A_n} &= (1, F_n - L_{n-1}) = (1, -F_{n-2}) \\ \overrightarrow{B_{n-1}A_{n+1}} &= (2, F_{n+1} - L_{n-1}) = (2, F_{n-3}) \\ \overrightarrow{A_nA_{n+1}} &= (1, F_{n+1} - F_n) = (1, F_{n-1}) , \end{aligned}$$

and so

$$\begin{aligned} \left|\overrightarrow{B_{n-1}A_n}\right| &= \sqrt{1 + F_{n-2}^2} \\ \left|\overrightarrow{B_{n-1}A_{n+1}}\right| &= \sqrt{4 + F_{n-3}^2} \\ \left|\overrightarrow{A_nA_{n+1}}\right| &= \sqrt{1 + F_{n-1}^2} . \end{aligned}$$

We can then calculate the general formula

$$\cos\theta_n = \cos\angle(A_n B_{n-1} A_{n+1}) = \frac{\overrightarrow{B_{n-1}A_n} \cdot \overrightarrow{B_{n-1}A_{n+1}}}{\left|\overrightarrow{B_{n-1}A_n}\right|\left|\overrightarrow{B_{n-1}A_{n+1}}\right|} = \frac{2 - F_{n-2}F_{n-3}}{\sqrt{1+F_{n-2}^2}\sqrt{4+F_{n-3}^2}}\,.$$

The following result represents a geometric interpretation of the relation between the Lucas and Fibonacci sequences:

Theorem 2. *The following relations in the triangle $\Delta B_{n-1}A_nA_{n+1}$ hold for $n \geq 2$:*

(a) $\theta_n = \angle(A_nB_{n-1}A_{n+1})$ *increases as n increases.*

(b) $\lim_{n\to\infty} \theta_n = \pi$.

(c) The quotients of the edge lengths of the triangle $\Delta B_{n-1}A_nA_{n+1}$ converge to φ and φ^2:

$$\lim_{n\to\infty} \frac{\left|\overrightarrow{B_{n-1}A_n}\right|}{\left|\overrightarrow{B_{n-1}A_{n+1}}\right|} = \varphi\,, \qquad \lim_{n\to\infty} \frac{\left|\overrightarrow{B_{n-1}A_n}\right|}{\left|\overrightarrow{B_{n-1}A_{n+1}}\right|} = \varphi\,, \quad \textit{and} \quad \lim_{n\to\infty} \frac{\left|\overrightarrow{A_nA_{n+1}}\right|}{\left|\overrightarrow{B_{n-1}A_{n+1}}\right|} = \varphi^2\,.$$

Proof.
(a) Since F_n increases as n increases,

$$\cos\theta_n = \frac{2 - F_{n-2}F_{n-3}}{\sqrt{1+F_{n-2}^2}\sqrt{4+F_{n-3}^2}} \geq \frac{2 - F_{n-1}F_{n-2}}{\sqrt{1+F_{n-1}^2}\sqrt{4+F_{n-2}^2}} = \cos\theta_{n+1}\,,$$

which implies that θ_n increases as n increases.
(b)

$$\lim_{n\to\infty}\cos\theta_n = \lim_{n\to\infty}\frac{2 - F_{n-2}F_{n-3}}{\sqrt{1+F_{n-2}^2}\sqrt{4+F_{n-3}^2}} = \lim_{n\to\infty}\frac{\frac{2}{F_{n-2}F_{n-3}} - 1}{\sqrt{\frac{1}{F_{n-2}^2}+1}\sqrt{\frac{4}{F_{n-3}^2}+1}} = -1\,,$$

so $\lim_{n\to\infty}\theta_n = \pi$.
(c) Since $\varphi \approx 1.618 > 1$, Binet's Formula (1) implies that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}\left(\varphi^{n+1} - \varphi^{-(n+1)}\right)}{\frac{1}{\sqrt{5}}\left(\varphi^{n} - \varphi^{-n}\right)} = \frac{\varphi^{2n+1} - \varphi^{-1}}{\varphi^{2n} - 1} = \varphi\,.$$

This limit, discovered by the great astronomer Johannes Kepler four centuries ago, implies that

$$\lim_{n\to\infty} \frac{\left|\overrightarrow{B_{n-1}A_n}\right|}{\left|\overrightarrow{B_{n-1}A_{n+1}}\right|} = \lim_{n\to\infty}\sqrt{\frac{1+F_{n-2}^2}{4+F_{n-3}^2}} = \lim_{n\to\infty}\sqrt{\frac{\frac{1}{F_{n-3}^2} + \left(\frac{F_{n-2}}{F_{n-3}}\right)^2}{\frac{4}{F_{n-3}^2}+1}} = \lim_{n\to\infty}\frac{F_{n-2}}{F_{n-3}} = \varphi\,.$$

The two remaining ratio limits are proved similarly. □

4 Final Remarks

We end this paper by recommending the book of Gielis [2], in which types of symmetries related to the Fibonacci sequence, observed in plants and shells, are nicely presented and illustrated.

Acknowledgements

The author is very indebted to the Editor in Chief of *Parabola* (UNSW), Dr Thomas Britz, for giving valuable suggestions for improving the contents and presentation of the paper.

References

[1] T. Ghose, What is the Fibonacci sequence?, Live Science, October 24, 2018, https://www.livescience.com/37470-fibonacci-sequence.html, last accessed on 24-05-2020.

[2] J. Gielis, *Inventing the Circle. The Geometry of Nature*, Geniaal, Antwerpen, 2003.

[3] Lucas number, *Wikipedia*, https://en.wikipedia.org/wiki/Lucas_number, last accessed on 24-05-2020.

[4] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Publ., Mineola, New York, 2008.

[5] Fibonacci numbers, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A000045, last accessed on 24-05-2020.

[6] Lucas numbers, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A000032, last accessed on 24-05-2020.

[7] Lucas number, *Wikipedia*, https://en.wikipedia.org/wiki/Lucas_number, last accessed on 24-05-2020.